The Search for the Cyclic Sieving Phenomenon in Plane Partitions

William J. Asztalos
DePaul University, wjasztalos@gmail.com

The Search for the Cyclic Sieving Phenomenon in Plane Partitions

Acknowledgements

The Search for the Cyclic Sieving Phenomenon in Plane Partitions

William Asztalos[1]

Department of Mathematical Sciences

T. Kyle Petersen, PhD; Faculty Advisor

Department of Mathematical Sciences

ABSTRACT The efforts of this research project are best understood in the context of the subfield of dynamical combinatorics, in which one enumerates a set of combinatorial objects by defining some action to guide the search for underlying structures. While there are many examples with varying degrees of complexity, the necklace problem, which concerns the possible unique configurations of beads in a ring up to rotational symmetry, is a well-known example. Though this sort of approach to enumeration has been around for a century or more, activity in this area has intensified in the last couple of decades. Perhaps the most startling development was the discovery of the cyclic sieving phenomenon, in which polynomial generating functions produce information about the sizes of rotational symmetry classes of objects. This technique is an extension of the "$q = -1$" phenomenon which classifies objects on the basis of being a fixed point or an element of a "mirrored" pair. In this study, we are on the hunt for rotational symmetries in plane partitions, with the ultimate goal of recovering the "magic" polynomial that will allows us to count the symmetry classes of these objects. The unique characteristics of plane partitions under our devised operation portend that attaining such a goal is feasible.

INTRODUCTION

This project involves a subfield of mathematics known as dynamic combinatorics. In this particular subfield, the conventional methodology is to begin with a set of mathematical objects that one wishes to study, with the question of "how many of these objects are there?" being the typical inquiry. Thereafter, one generally defines some kind of action on the set so as to uncover underlying structure or symmetries in the objects in the set. What's more, we can actually determine the number and size of the symmetry classes of the set under the operation in order to determine the size of the original set itself, since these symmetry classes partition the set in a natural fashion. As it happens, this approach is centuries old, and there are plenty of rich examples showcasing how approaching the task of enumeration through a dynamic combinatorial lens can greatly simplify incredibly complex problems. As an example of

---

[1] *wjasztalos@gmail.com*

this process, we may consider the task of enumerating binary strings of length 5. An immensely uninspiring way to do this would be to simply generate all of them, as in Figure 1. Not only does this task fail to reveal the underlying structure of these objects, but it also becomes increasingly difficult when we seek to obtain the number of binary strings of length n for a given natural number n.

11110	11101	11011	10111
01111	11100	11010	10110
01110	11001	10101	10011
01101	01011	00111	11000
10100	01100	10010	10001
01010	01001	00110	00101
00011	10000	01000	00100
00010	00001	11111	00000

Figure 1

Enumerating these binary strings may be greatly simplified by devising a cyclic action on this set, however. In particular, if we utilize the familiar operation of cyclic permutation, T, which is performed by taking the first entry of a binary string and moving it to the end of the string, the binary strings of length 5 may be partitioned into symmetry classes as shown in Figure 2 below. In this figure, the columns are the symmetry classes the binary strings fall into under T, with the image of a binary string under this operation being the binary string below it (and we naturally return to the top after reaching the last one).

11110	11100	11000	10000	11010
11101	11001	10001	00001	10101
11011	10011	00011	00010	01011
10111	00111	00110	00100	10110
01111	01110	01100	01000	01101

11111	10100	00000
11111	01001	00000
11111	10010	00000
11111	00101	00000
11111	01010	00000

Figure 2

What's more, we can represent each of these symmetry classes as what combinatorialists call a binary necklace by taking a representative binary string, replacing an entry of 1 with a black bead, and an entry of 0 with a white bead, stringing the beads together and tying the ends together. For instance, we may express the orbit of the binary string 11010 as a binary necklace as shown in Figure 3. Observe that if we establish a canonical way of reading these necklaces, starting at the top and proceeding counterclockwise for instance, then rotating the necklace clockwise (as indicated by the arrow) will generate all the other binary strings in the orbit.
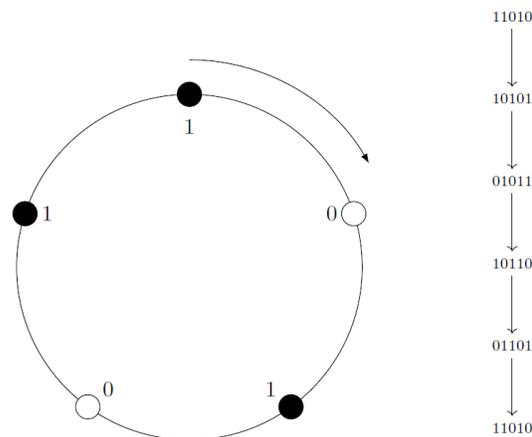

Figure 3

At this juncture, we have already substantially simplified the process of enumerating binary strings of length 5; in order to further reduce the difficulty of this task, we turn to the cyclic sieving phenomenon, a brand new combinatorial "gadget" of sorts.

THE CYCLIC SIEVING PHENOMENON

The discovery of the cyclic sieving phenomenon resulted from many explorations in the field of dynamic combinatorics in the past two decades, particularly by observing the results of supplying certain values to these incredibly helpful polynomials combinatorialists call generating functions. For a set of combinatorial objects, X, that one wishes to study, a generating function is a polynomial $X(q) = \sum_{x \in X} q^{s(x)}$ which encodes statistics for elements of the set. Naturally, if one calculates $X(1)$, the result gives the number of elements in our set, X. After experimenting with several sets of combinatorial objects, generating functions for the set, and various actions on the set, combinatorialists determined that the

evaluation of $X(-1)$ indicates whether the objects of X have "mirror" symmetry for some action on X. This is known in the literature as the "$q = -1$" phenomenon. As we observed with binary strings and their orbits represented as necklaces, there are more rotations possible than $180°$ rotation (which may be thought of as analogous to the aforementioned "mirror" symmetry). A natural question, then, is if there are other symmetries possible aside from mirror symmetry—can we have n-fold symmetries of some kind, e.g., n-fold rotational symmetry?

As it happens, the cyclic sieving phenomenon provides one approach to the preceding question. To check for k-fold symmetries in a set of combinatorial objects, we plug the value ζ^k into $X(q)$, where $\zeta = e^{\frac{2\pi i}{n}}$. This complex number is a so-called "n^{th} root of unity". It corresponds to rotation by an angle of $\frac{2\pi}{n}$ radians. If $X(\zeta^k)$ is a positive integer, m, then the cyclic sieving phenomenon affirms that there exists some kind of action on the set X such that applying said operation k times fixes m of the elements of X. We may note that if $n = 2$, then $\zeta = -1$, thus alluding to the "$q = -1$" phenomenon aforementioned. It is especially interesting that plugging ζ, which is a complex number, into a polynomial with real coefficients would somehow output a positive integer, and one describing underlying symmetry in the original set at that.

Presently, we may return to the example of enumerating binary strings to see just how the cyclic sieving phenomenon works in practice. This time, we will consider the binary strings of length 6. In order to construct a generating function for this set of objects, we will need to count some kind of statistic for objects in this set. A perfectly suitable choice is a statistic called the major index of a binary string, which is the total of the relative positions of an entry of 1 immediately followed by an entry of 0. As an example, the binary string 010010 has a major index of 7 since there is an entry of "10" in both the second and fifth position. Another potential way to distribute binary strings of length 6 according to some statistic is according to the

area bounded by their expression as a lattice path. By encoding an entry of 1 as a movement northward and an entry of 0 as a step eastward, we can create a bijection between binary strings of length 6 and walks in a lattice of dimensions corresponding to the number of 0 and 1 entries. For instance, the binary string 001001 corresponds to the lattice path in Figure 4, with this lattice path possessing a bounded area of 6, as indicated in Figure 5.
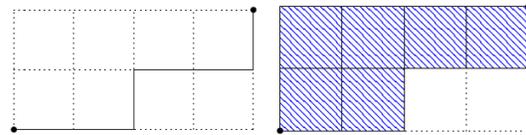

Figure 4   **Figure 5**

As it happens, counting binary strings according to either of these statistics yields the same exact generating function. The details of proving this equivalence are provided in the sixth chapter of Petersen's *Eulerian Numbers* [2]; it is a consequence of [1, Theorems 6.2 and 6.4]. In any case, we obtain the following generating function for binary strings of length 6:

$$X(q) = q^9 + 3q^8 + 4q^7 + 7q^6 + 9q^5 + 11q^4 + 9q^3 + 8q^2 + 5q + 7 \quad (1)$$

Now, upon plugging in powers of $\zeta = e^{\frac{2\pi i}{6}} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, we find that $X(\zeta) = 2$, $X(\zeta^2) = 4$, $X(\zeta^3) = 8$, $X(\zeta^4) = 4$, $X(\zeta^5) = 2$, $X(1) = 64$. By analyzing these results carefully, we were able to conclude that there are precisely two fixed points, one 2-cycle, two 3-cycles, no 4-cycles or 5-cycles, and nine 6-cycles. Altogether, then, there are $2 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 + 0 \cdot 4 + 0 \cdot 5 + 9 \cdot 6 = 64$ binary strings of length 6. Now we have seen first hand how the cyclic sieving phenomenon can simplify the process of combinatorial enumeration! Digging a tad further, we may make another fascinating observation. If we examine the binary necklace corresponding to 010010, an element of a 3-cycle in our set, we find that the necklace itself has symmetry by a rotation of $\frac{2\pi 3}{6} = \pi$ radians about its center, which corresponds to ζ^3, as illustrated in Figure 6.
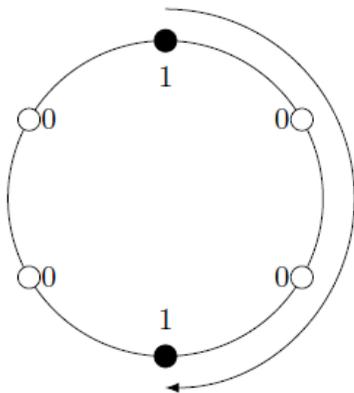
Figure 6

Employing our bijection with lattice paths, we can also construct the figure depicting the same 3-cycle containing 010010 included below. We may observe that performing cyclic permutation on these binary strings becomes equivalent to rotating the entirety of Figure 7 by $\frac{2\pi}{3}$ radians clockwise; if we focus on one lattice path on the graph below, the lattice path which occupies this same position after rotating the entire figure by $\frac{2\pi}{3}$ radians about its center is the image of the original lattice path under cyclic permutation. This illustrates yet another connection between rotational symmetry and the latent symmetries of binary strings under cyclic permutation.
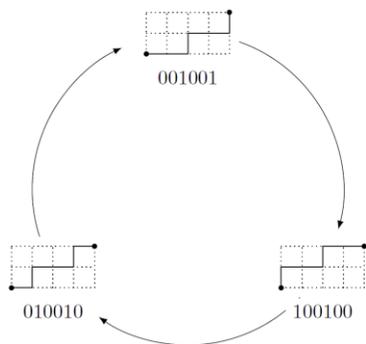

Figure 7

All of the preceding is well and good, but how exactly does it pertain to our research project? More importantly, how can we use the cyclic sieving phenomenon to count the number of plane partitions?

PLANE PARTITIONS

The particular goal of this study was to enumerate the number of plane partitions of an integer, which may be thought of as a generalization of the more familiar integer partitions. In particular, a *plane partition* is a two-dimensional array of nonnegative integers whose sum is n, such that the columns and rows are nonincreasing. This gives an informal understanding of plane partitions, whereas the following definition makes that intuition more precise.

Definition 1 (Plane Partition). A plane partition of a nonnegative integer n is an array, $\lambda = \left(\lambda_{i,j}\right)$ for $1 \leq i,j \leq n$ in which $0 \leq \lambda_{i,j} \leq n$, $\lambda_{i,j} \geq \lambda_{i,j+1}$, and $\lambda_{i,j} \geq \lambda_{i+1,j}$ for all $1 \leq i,j \leq n$, and for which $\sum \lambda_{i,j} = n$. The array in Figure 8 is an example of a plane partition of 33.

$$\lambda = \begin{pmatrix} 9 & 3 & 2 & 1 & 0 & 0 & & 0 \\ 7 & 3 & 1 & 0 & 0 & & & 0 \\ 2 & 2 & 0 & 0 & & & \cdots & 0 \\ 2 & 0 & 0 & & & & & 0 \\ 1 & 0 & & & & & & 0 \\ 0 & & & & & \ddots & & 0 \\ & & & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Figure 8

Given that a lot of these entries are zeroes, we may discard the zero entries to express the plane partition in a more compact manner, as shown below in Figure 9. The boxes in which the numbers are written form what is known as a *Young diagram*.

Figure 9

One other way to represent a plane partition of n is as a stack of cubes, with each nonzero entry of the array indicating a stack of unit boxes with a height equivalent to the value of the entry in the

array. In Figure 10, we express our running example, λ, from Figure 9 as a box piling.
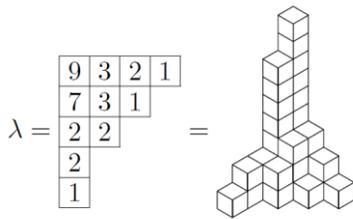

Figure 10

Exploring the properties of plane partitions has occupied combinatorialists since the days of MacMahon [1]. Indeed, while some very particular symmetries of plane partitions have been studied in the past, the rather unruly and unpredictable nature of plane partitions has prevented the study of more generalized underlying symmetries in these objects. Their complexity has made just counting them a trying task as well. To give some perspective, there are 118,794 plane partitions of 21; this is quite a large number of objects considering 21 is one of the smaller cases! Fortunately, the cyclic sieving phenomenon will be able to assist us in this task. Before we can utilize the cyclic sieving phenomenon, however, we must devise an operation on the set of plane partitions.

ROTATE AND RECTIFY

Initially, we had hoped to uncover an operation on plane partitions that would preserve the number of boxes of a plane partition when expressed as a box piling (as this would preserve the actual number the entries total to as a result). Unfortunately, any operations that satisfied this requirement often failed others, oftentimes taking us out of the set of plane partitions, or failing to be cyclic altogether. We did however, develop some desirable operations which preserved the "dimensions" of a plane partition, which is to say that it carried an $A \times B \times C$ plane partition to another $A \times B \times C$ plane partition.

Definition 2 (Plane Partitions in a Box). Let a, b, and c be nonnegative integers. Define $\Lambda_{a,b,c}$ to be the set of those plane partitions $\lambda = (\lambda_{i,j})$ with each entry $\lambda_{i,j} \leq c$ and $\lambda_{i,j} = 0$ if $i > a$ or $j > b$.

Thus, an $A \times B \times C$ plane partition is one whose cube piling representation can be contained in an $A \times B \times C$ box. Working in the context of $A \times B \times C$ plane partitions, we were able to develop one operation which showed special promise, which we call "Rotate and Rectify". In essence, this operation is an extension of the cyclic permutation action we defined for binary strings. The actual details of how this operation works are unimportant; what is particularly noteworthy is that Rotate and Rectify carries $A \times B \times C$ plane partitions to $A \times B \times C$ plane partitions, and that plane partitions either begin in a cycle or fall into a cycle under this operation. The only caveat is that, like the aforementioned operations, Rotate and Rectify fails to preserve the actual total of the plane partition (i.e., the number of boxes when expressed as a cube piling). Figure 11 is an example of a cycle under this action. Note the similarity to the 3-cycle of binary strings in Figure 7.
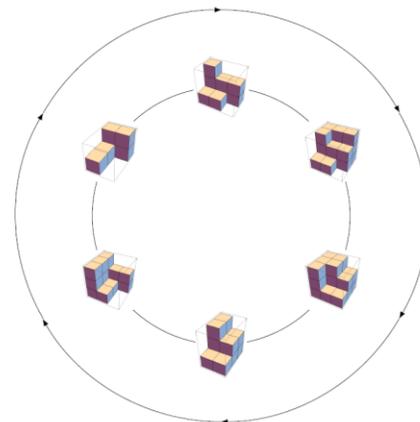

Figure 11

Since the set of $A \times B \times C$ plane partitions is finite, these cycles have a way of naturally partitioning this set, in the same way that the symmetry classes partitioned the set of binary strings into orbits. Now, it would be wise to see just how Rotate and Rectify divides up the set of $A \times B \times C$ plane partitions. Employing Mathematica, we can create a program to perform Rotate and Rectify on plane partitions, and draw a graph with arrows between an input plane partition and its image under this action. Figure 12 is a graph of the $3 \times 3 \times 3$ case.
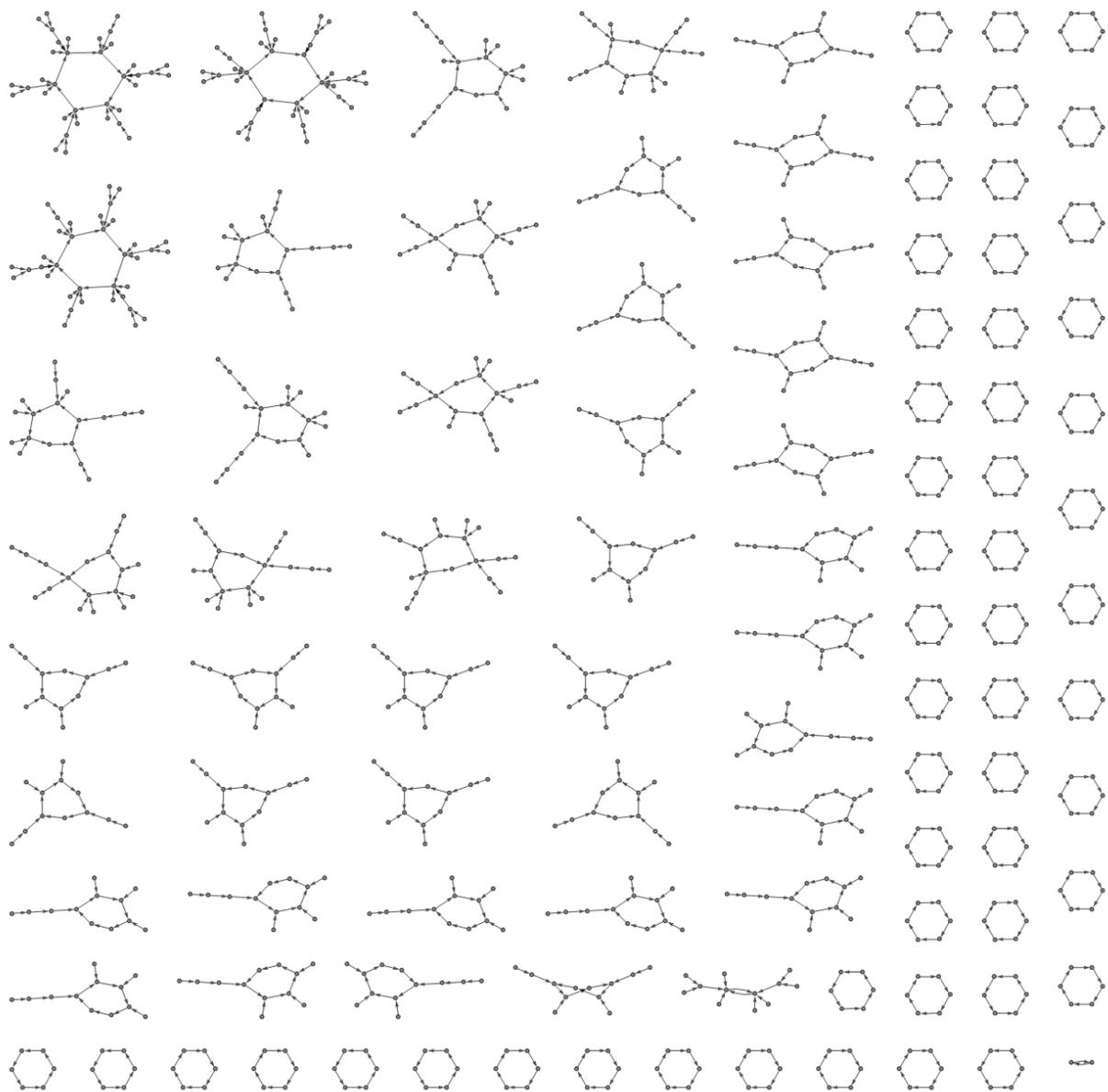
Figure 12: A graph of the cycles of $3 \times 3 \times 3$ plane partitions under rotate and rectify.

Each of these little nodes in the above graph is a $3 \times 3 \times 3$ plane partition, and, if you look closely, you can see a small arrow indicating which plane partition maps to which under Rotate and Rectify. This kind of structure alone suggests some kind of underlying symmetry in these objects, and understanding it would naturally prove instrumental to understanding the finer properties of plane partitions themselves. In fact, if we count the number of elements lying in cycles, and those lying outside of cycles in the $2 \times 2 \times C$ case, we obtain the following table of values.

c	$|X|$	$|X^c|$
1	6	0
2	19	1
3	44	6
4	85	20
5	146	50
6	231	105
7	344	196
⋮	⋮	⋮

Table 1

Upon plugging these values into the Online Encyclopedia of Integer Sequences, we find that the number of plane partitions in cycles is counted by the octahedral numbers, and that the number of elements outside of cycles is counted by the 4-D pyramidal numbers, both of which are fairly well-studied sequences. The fact that this bijection is possible suggests that Rotate and Rectify is unearthing some of the more latent symmetries in plane partitions, implying that we are on the right path! Indeed, pushing a little further, we found that Rotate and Rectify was an appropriate tool for the job, leading to some interesting conjectures and nice results.

RESULTS AND DIRECTIONS FOR FURTHER STUDY

At this juncture, we have determined the cyclic structure of some of the smaller cases of $A \times B \times C$ plane partitions, with indication that our approach may be extended and used in the general case. Moreover, we have uncovered some symmetries in the actual cycle structure itself relating to the dimensions of the box plane partitions are contained in. Finally, bijections have been constructed with the cyclic elements of certain cases and well-known sequences of integers such as the octahedral numbers, as aforementioned.

As for what lies ahead, there still remains the task of proving the more general cases, which, admittedly, will prove a more trying task. Moreover, across the way we have uncovered potential connections to the disciplines of analysis and topology which would likely merit further study, and may even furnish the proof of the more general cases we seek. Perhaps most important will be the discovery of the actual generating function for the set of $A \times B \times C$ plane partitions, which will actually allow us to use the Cyclic Sieving Phenomenon to its full effect. Rest assured, we have begun the search for potential candidates for this generating function. Moreover, both Dr. Petersen and I intend to pursue this topic much further; over just a few months in the summer we uncovered quite a bit of the hidden nature of plane partitions, and we have really just broken the surface. Time will tell what manner of secrets we can uncover in these fascinating objects.

ACKNOWLEDGEMENTS

REFERENCES

[1] P. A. MacMahon. *Partitions of Numbers Whose Graphs Possess Symmetry*. Cambridge Philosophical Society, 1898.

[2] T. K. Petersen. *Eulerian Numbers*. Springer: Birkhäuser Advanced Texts Basler Lehrbücher, New York, 2015.