2013

Max-Min Up-Down Permutations

Fiacha Heneghan
fdheneghan@gmail.com

Ashley Sliva
ashleyanna2@gmail.com

Follow this and additional works at: https://via.library.depaul.edu/depaul-disc

Part of the Physical Sciences and Mathematics Commons

Acknowledgements
Faculty Advisor: T. Kyle Petersen, Department of Mathematical Sciences

Recommended Citation
Available at: https://via.library.depaul.edu/depaul-disc/vol2/iss1/1

This Article is brought to you for free and open access by the College of Science and Health at Via Sapientiae. It has been accepted for inclusion in DePaul Discoveries by an authorized editor of Via Sapientiae. For more information, please contact wsulliv6@depaul.edu, c.mcclure@depaul.edu.
A QUESTION ABOUT UP-DOWN PERMUTATIONS

A permutation is a one-to-one and onto map (function) from an ordered set to itself. We will generally consider permutations of the set \{1, 2, …, n\}. The permutations of \{1, 2, 3\} are, for example, 123, 132, 213, 231, 312, 321. These are abbreviated representations of the permutations of \{1, 2, 3\}, where 132, for instance, represents the function \( f \) such that \( f(1) = 1 \), \( f(2) = 3 \), \( f(3) = 2 \). Note that for any \( n \) there are a possible \( n! \) distinct permutations.

We are interested in a particular kind of permutation known as an alternating permutation. A permutation, \( w = w_1 w_2 … \) is called alternating if either \( w_1 > w_2 < w_3 > w_4 < … \), in which case we call it down-up alternating, or \( w_1 < w_2 > w_3 < w_4 > … \), which we call up-down alternating. For example, an up-down permutation of 7 is 3517264, and an example of a down-up permutation is 5317624. See Richard Stanley’s paper for a recent survey of research on alternating permutations [4].

ABSTRACT

In this note we study a curious refinement of the Euler numbers, \( E_n \), which count the number of “up-down” permutations of length \( n \). Namely, we define two sequences \( E_{n}^{\text{min-max}} \) and \( E_{n}^{\text{max-min}} \) (counting up-down permutations according to whether the digit 1 occurs before or after \( n \)) such that \( E_n = E_n^{\text{min-max}} + E_n^{\text{max-min}} \). We show \( E_{2n}^{\text{min-max}} - E_{2n-1}^{\text{max-min}} = 0 \), and using generating function techniques show that \( E_{2n}^{\text{min-max}} - E_{2n}^{\text{max-min}} = E_{2n-2} \). We are then able to show that the ratio \( \frac{E_n^{\text{min-max}}}{E_n} \) has limit 1.

There is a simple correspondence (a one-to-one and onto map) between up-down alternating and down-up alternating permutations: for any up-down permutation of \{1, 2, …, n\} replacing the letter \( k \) with \( n+1-k \) will yield a corresponding down-up permutation (and vice-versa). Notice that the permutations 351724 and 5371624 can be obtained from each other in this way.

Let \( E_n \) denote the number of up-down (respectively, down-up) permutations of \( n \). Thus \( 2E_n \) is the number of all alternating permutations. For example, \( E_1 = 1 \), \( E_2 = 1 \), \( E_3 = 2 \), and \( E_4 = 5 \). Larger values can be seen in Table 2. These are known as the Euler numbers, which have been well-studied. See, for example, David Callan’s paper on down-up permutations and 0-1-2 trees for another interesting combinatorial interpretation for these numbers [2]. We will focus, however, on permutations.
Another characteristic of permutations, suggested in Callan’s work [2], is the relative placement of 1 and n. In particular, reading a permutation left to right, we are concerned with whether 1 precedes n or vice-versa. If 1 appears before n, we call the permutation min-max; if it follows n, we say it is max-min.

There is a correspondence between min-max alternating permutations and max-min alternating permutations through reversing the permutation. For example, the min-max permutation 153624 becomes the max-min permutation 426351, while 3517264 becomes 4627153. (Notice that this reversal does not preserve up-downness in the even length case, but does in the odd. Observe, also, that reversal gives a correspondence between up-down min-max and up-down max-min permutations when n is odd.) Thus, there are the same number of min-max alternating permutations as there are max-min alternating permutations. Since the total number of alternating permutations is $2E_n$, this implies that $E_n$ counts the number of min-max alternating permutations.

<p>| TABLE 1 |
| Alternating permutations for $n = 4$ and $n = 5$ organized by the properties max-min/min-max and up-down/down-up. |</p>
<table>
<thead>
<tr>
<th>max-min</th>
<th>min-max</th>
<th>up-down</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>35241</td>
</tr>
<tr>
<td></td>
<td></td>
<td>45231</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35142</td>
</tr>
<tr>
<td></td>
<td></td>
<td>45132</td>
</tr>
<tr>
<td></td>
<td></td>
<td>45132</td>
</tr>
<tr>
<td></td>
<td></td>
<td>34251</td>
</tr>
<tr>
<td></td>
<td></td>
<td>24351</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35241</td>
</tr>
<tr>
<td></td>
<td></td>
<td>35142</td>
</tr>
<tr>
<td></td>
<td></td>
<td>23144</td>
</tr>
<tr>
<td></td>
<td>2413</td>
<td>1324</td>
</tr>
<tr>
<td></td>
<td>3412</td>
<td>1423</td>
</tr>
<tr>
<td></td>
<td>2314</td>
<td></td>
</tr>
<tr>
<td>down-up</td>
<td>4231</td>
<td>3142</td>
</tr>
<tr>
<td></td>
<td>3241</td>
<td>2143</td>
</tr>
<tr>
<td></td>
<td>4132</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4231</td>
<td>3142</td>
</tr>
<tr>
<td></td>
<td>3241</td>
<td>2143</td>
</tr>
<tr>
<td></td>
<td>4132</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4231</td>
<td>3142</td>
</tr>
<tr>
<td></td>
<td>3241</td>
<td>2143</td>
</tr>
<tr>
<td></td>
<td>4132</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4231</td>
<td>3142</td>
</tr>
<tr>
<td></td>
<td>3241</td>
<td>2143</td>
</tr>
<tr>
<td></td>
<td>4132</td>
<td></td>
</tr>
<tr>
<td></td>
<td>4231</td>
<td>3142</td>
</tr>
<tr>
<td></td>
<td>3241</td>
<td>2143</td>
</tr>
<tr>
<td></td>
<td>4132</td>
<td></td>
</tr>
</tbody>
</table>

We now have two ways of evenly partitioning the set of alternating permutations of $n$: according to whether a permutation is up-down or down-up and according to whether it is min-max or max-min. Take a look at Table 1. We notice that for the case $n = 4$, there are five min-max permutations and five max-min permutations. Likewise, there are five up-down permutations and five down-up permutations. On the other hand, there are three up-down min-max permutations and three down-up max-min permutations, but only two up-down max-min permutations and two down-up min-max permutations. For the case $n = 5$, however, each of the four subsets has the same number of permutations, eight. In this paper, we will make precise the relationships between these four subsets of alternating permutations.

Let $E_n^{\text{min-max}}$ denote the number of up-down min-max permutations of $n$, and let $E_n^{\text{max-min}}$ denote the number of up-down max-min permutations of $n$. Notice that

$$E_n = E_n^{\text{min-max}} + E_n^{\text{max-min}}.$$

We include the first few values of these sequences and some related quantities in Table 2. The sequences $E_n^{\text{min-max}}$ and $E_n^{\text{max-min}}$ do not appear in the Online Encyclopedia of Integer Sequences [3] and thus are unlikely to have been studied in the literature.

<p>| TABLE 2 |
| Number of up-down min-max/min-max permutations up to $n = 10$. |</p>
<table>
<thead>
<tr>
<th>$n$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_n$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>16</td>
<td>61</td>
<td>272</td>
<td>1385</td>
<td>7936</td>
<td>50521</td>
</tr>
<tr>
<td>$E_n^{\text{min-max}}$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>33</td>
<td>136</td>
<td>723</td>
<td>3968</td>
<td>25953</td>
</tr>
<tr>
<td>$E_n^{\text{max-min}}$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>8</td>
<td>28</td>
<td>136</td>
<td>662</td>
<td>3968</td>
<td>24568</td>
</tr>
<tr>
<td>$E_n^{\text{min-max}} - E_n^{\text{max-min}}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>5</td>
<td>0</td>
<td>61</td>
<td>0</td>
<td>1385</td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>.737</td>
<td>1</td>
<td>.916</td>
<td>1</td>
<td>.947</td>
</tr>
</tbody>
</table>
**GENERATING FUNCTION BACKGROUND**

Readers who have taken calculus may remember the concept of a power series, an infinite sum, often of the form

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots \]

In calculus, series such as these are often used to approximate functions, but in combinatorics, they are frequently used as an index for a sequence. That is, it is often useful to consider a power series whose coefficients are the entries in a sequence we are interested in. In this case, we call a formal power series a generating function for a sequence. In this paper, we will be working with exponential generating functions, power series of the form

\[ f(z) = \sum_{n=0}^{\infty} \frac{a_n z^n}{n!} \]

where \(a_n\) is the \(n^{th}\) term in a given sequence of numbers.

The Belgian combinatorialist Desiré André was the first mathematician to prove that the sequence of Euler numbers have the exponential generating function

\[ \sec z + \tan z = \sum_{n=0}^{\infty} E_n \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{2z^3}{6} + \frac{5z^4}{24} + \frac{16z^5}{120} + \ldots \]

See [1, 4].

An interesting feature of this expression is that \(\sec z \) and \(\tan z \) are themselves the individual exponential generating functions for the sequences of even and odd Euler numbers respectively, so the generating function divides up neatly along these lines.

\[ \sec z = \sum_{n=0}^{\infty} E_{2n} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \ldots \]

and

\[ \tan z = \sum_{n=0}^{\infty} E_{2n+1} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{2z^3}{6} + \frac{16z^5}{120} + \ldots \]

These facts equip us to describe the behavior of our refinement of up-down permutations via min-max and max-min.

**MAIN RESULTS**

Using combinatorial reasoning, we were able to discover simple generating functions for \(E_{n}^{\text{min-max}}\) and \(E_{n}^{\text{max-min}}\), which represent refinements of the original generating function of the sequence counting simply the number of up-down permutations, \(\sec z + \tan z\). The following is our main result.

**Theorem 1.**

Let \(f(z)\) be the exponential generating function of \(E_{n}^{\text{min-max}}\) and \(g(z)\) be the exponential generating function of \(E_{n}^{\text{max-min}}\). Then

\[ f(z) = \sec^3 z + \sec^2 z \tan z \]

and

\[ g(z) = \sec^2 z \tan z + \sec z \tan^2 z. \]

Again, when we say that these are the generating functions for \(E_{n}^{\text{min-max}}\) and \(E_{n}^{\text{max-min}}\), we mean that the number of up-down min-max permutations and up-down max-min permutations of length \(n\) is indexed by the \(n^{th}\) term in the respective series expansions of these functions, namely:

\[ f(z) = \sum_{n=0}^{\infty} E_{n}^{\text{min-max}} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{8z^3}{6} + \frac{33z^4}{24} + \ldots \]

and

\[ g(z) = \sum_{n=0}^{\infty} E_{n}^{\text{max-min}} \frac{z^n}{n!} = z + \frac{z^2}{2} + \frac{8z^3}{6} + \frac{33z^4}{24} + \ldots \]

To gain some intuition about why these generating functions work, consider a generic up-down min-max permutation:

\[ \begin{array}{c}
\uparrow \\
\downarrow \\
\uparrow \\
\downarrow \\
\uparrow \\
\end{array} \]

Notice that from the first letter to the 1 we have an alternating permutation of even length (i.e. \(n\) is even), between the 1 and the \(n\) is an alternating permutation of even length, and finally after the \(n\) we have an alternating permutation that will be either even or odd depending on \(n\). Recall that the sequence counting even-length
alternating permutations (restricted to either up-down or down-up permutations only) is encoded by the generating function \( \sec z \) and the analogous sequence for odd-length permutations has the generating function \( \tan z \). Loosely speaking then, each of the three parts discussed of an up-down min-max permutation can be represented by \( \sec z \), \( \sec z \), and \( \sec z + \tan z \), respectively.

Generating a random up-down min-max permutation consists of choosing the location of 1 and then filling in the intervening spaces with appropriate alternating permutations of the remaining letters. Each of these permutations is created independently, so we should multiply the number of possibilities (again, loosely represented by either \( \sec z \), \( \sec z \), or \( \sec z + \tan z \)) for each part to get the number of total up-down min-max permutations. In this case, we have \( \sec z \) \( \sec z \) \( \sec z + \tan z \) = \( \sec z^3 + \sec z^3 \tan z \), which is our generating function \( f(z) \). The reasoning for the generating function \( g(z) \) is similar.

We can also show an interesting feature of these sequences that is hinted at in Table 2. First note that the sum of our functions is the second derivative of \( \sec z + \tan z \), as it should be, since

\[
\frac{d^2}{dz^2} [\sec z + \tan z] = \frac{d^2}{dz^2} \sum_{n \geq 0} E_n \frac{z^n}{n!} = \sum_{n \geq 0} E_{n+2} \frac{z^n}{n!}
\]

while

\[
f(z) + g(z) = \sum_{n \geq 0} \left( E_{n+2}^{\min\text{-}\max} + E_{n+2}^{\max\text{-}\min} \right) \frac{z^n}{n!}
\]

This is the generating function analogue of the identity \( E_n = E_n^{\min\text{-}\max} + E_n^{\max\text{-}\min} \) (for \( n \geq 2 \)). Similarly, we can check that the difference of the two functions has the same derivative as \( \sec z \), giving a result for which we have no simple combinatorial explanation.

**Corollary 2.**
We have

\[
\frac{d}{dz} [f(z) - g(z)] = \frac{d}{dz} \sec z,
\]

and thus for \( n \geq 1 \),

\[
E_{2n}^{\min\text{-}\max} - E_{2n}^{\max\text{-}\min} = E_{2n-2}.
\]

In another direction, we are also able to prove that the ratio of min-max up-down permutations to max-min up-down permutations approaches 1. The last line of the table in Table 2 gives some empirical evidence that \( E_{n+2}^{\max\text{-}\min} \) and \( E_{n+2}^{\min\text{-}\max} \) are indeed getting closer in value.

**Corollary 3.**
We have

\[
\lim_{n \to \infty} \frac{E_{2n}^{\max\text{-}\min}}{E_{2n}^{\min\text{-}\max}} = 1.
\]

For odd integers, the ratio is exactly 1. For even integers, Corollary 2 implies that it suffices to show

\[
\lim_{n \to \infty} \frac{E_{2n-2}}{E_{2n}} = 0.
\]

The proof for this fact relies on a recurrence for the Euler numbers and the use of some limit techniques from calculus.

**Methods**
Researching alternating permutations and various statistics, characteristics, and measurements of permutations in general led to combining the idea of up-down permutations along with the max-min statistic. We began by counting the small cases of \( n \) by hand and searching the data for any observable patterns. From here we began making conjectures which we tested using the software Maple, a computer algebra system. Maple allowed us to verify our conjectures by gathering data for larger cases of \( n \) which would not be possible to work out by hand. Once we were reasonably confident in our hypotheses, we set about proving them using techniques from combinatorics and analysis.

---

https://via.library.depaul.edu/depaul-disc/vol2/iss1/1
IMPLICATIONS AND SIGNIFICANCE

Through our research we discovered two new sequences of integers and a generating function for each of these sequences, found a correspondence to the well-known Euler Numbers, and proved several relationships between these two sequences. The Euler Numbers are an ongoing area of interest in combinatorial research and our findings expand on ideas and theorems behind the Euler Numbers and provide a new combinatorial interpretation for the significance of the sequence of Euler numbers. Combinatorics itself is an area of mathematics that has well-known applications in computer science, probability theory, geometry, and statistical physics.
REFERENCES


[2] D. Callan, A Note on Downup Permutations and 0-1-2 Trees


Many students present their research to fellow students, professors and community members at the 9th Annual Natural Sciences, Mathematics, and Technology Showcase.