Spring 6-10-2022

Opinion Profile Dynamics with Uniform Bounded Confidence: Analysis & Simulations

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By

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A Dissertation
Submitted to Department of Mathematical Sciences
DePaul University
In Partial Fulfillment of the Requirements
for the Degree of Master of Science
June 2022

Acknowledgements

I would like to first thank my advisor, Dr. Ugarcovici who brought clarity, ingenuity, and enjoyment to this work. Never have I experienced a teacher as dedicated as you. I want to thank my brother Matthew for constantly listening to my rants and having the patience of a saint. I would also like to thank my sister Jocelyn for her words of encouragement and her late-night cookies while working through problems. Last but certainly not least, I want to thank my parents Nancy and Fidel for their tremendous support and love. Words cannot express how grateful I am for everything you guys have done that has led me to this point.

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Abstract

An opinion profile can describe a set of individuals, groups or entities that form an opinion over a topic. Given a number of agents, a time evolving opinion denoted by a vector $\vec{x}(t)$ can be described over time using the linear model $\vec{x}(t+1) = A\big(t, \vec{x}(t)\big)\vec{x}(t)$ where t is a time variable and the matrix A (which might depend on time and the current profile) contains the weights that affect the dynamics of profile $\vec{x}(t)$. With varying conditions over time, these profiles can form groups, polarize, fragment, or reach consensus. In this work, we investigate the uniform bounded confidence model: an agent adjusts their opinion over time by averaging the opinions of those agents which are ε-close to that agent. By performing Monte Carlo simulations, we observed the phenomenon that consensus is very unlikely for $\varepsilon < 0.2$ and becomes prevalent as $\varepsilon \geq 0.2$. To better understand this behavior, we conduct an in-depth analysis of evenly spaced profiles. We prove that $\varepsilon = 0.35$ guarantees consensus of all evenly spaced profiles for an odd number of agents $N \geq 5$. Numerical simulations also suggest that a smaller threshold value of $\varepsilon = 0.2$ guarantees consensus for these same profiles if $N \geq 17$. Also, for an even number of agents, numerical simulations show that $\varepsilon = 0.25$ seems to guarantee consensus if $N \geq 26$. Most importantly, our research indicates that the formation of separate groups created after 8-time steps is enough to determine whether consensus or fragmentation occurs. This mitigates a significant increase in processing time for certain values of ε within this bounded confidence algorithm.

Chapter 1: Introduction and Background

Opinion dynamics is the study of opinions using mathematical models, dynamical systems and stochastic processes. Recently, a developed method called bounded confidence has introduced new advancements in this field. The model was thoroughly investigated by Hegselmann and Krause in their 2002 article “Opinion Dynamics and Bounded Confidence Models, Analysis, and Simulation” [5]. This model is especially interesting because of its mathematical rigor, flexibility of application and focus on uniquely describing opinion profiles through vector analysis and discrete time.

This literature seeks to expand upon opinion dynamics using bounded confidence. First, we investigate the significance of the bounded confidence value ε and its relationship to consensus or fragmentation. Then, using Monte Carlo Simulations we would attempt to identify patterns of this probabilistic system. Most importantly, this research would conduct an in-depth analysis of evenly spaced opinion profiles due to its potential relation to this random system. Before these goals can be accomplished, it is important to describe an opinion profile in terms of vectors.

In this context, an opinion profile is defined as a vector with N number of entries x_i each being referred to as an “agent”. An agent described this way can be represented as an individual, a group, or an entity that can form an opinion about a particular subject. Each agent consists of a probabilistic number such that $0 \leq x_i \leq 1$ for each agent, where $1 \leq i \leq N$. For example, an initial profile studying the opinions of 10 people (or 10 agents) in favor of vaccinations may look like the following set:

$$\vec{x} = [0.29, 0.113, 0.97, 0.84, 0.57, 0.64, 0.52, 0.059, 0.33, 0.42]$$

An agent written this way can only have a proclivity towards yes (being close to 1) and no (being close to 0). With this opinion profile we can now introduce $\vec{x}$ as a function of time t.

Classical (time-independent) model. The immediate extension of the above idea is referred to as the classical model (CM) from the research of DeGroot [2]. In this model, the dynamics of the opinion profile is given by

$$\vec{x}(t+1) = A\vec{x}(t), t \in \{0, 1, 2, \dots\}.$$

Put simply, the model states that the next time step in the opinion profile, $\vec{x}(t+1)$ is obtained from the profile $\vec{x}(t)$ adjusted by a row-stochastic matrix A consisting of the weights of all opinions.

We say that an initial profile $\vec{x}(0) \in [0,1]^N$ has the *consensus property* if there exists $c \in [0,1]$ such that $\lim_{t\to\infty} x_i(t) = c$ for all $i \in 1, \ldots, N$. Consensus is reached when the opinion of each agent x_i as $t \to \infty$ is equal to the same constant "c" [2]. This discrete linear model (with constant weights) is well-understood by using the classical properties of row-stochastic matrices and Perron-Frobenius theory [9]. For example, if for any two $i, j \in 1, \ldots, N$ there exists some $k \in 1, \ldots, N$ such that $a_{ik} > 0$ and $a_{jk} > 0$ (i.e., matrix A is irreducible) then the consensus property holds [2]. In this context, this property is also called the "third agent principle". In addition, the consensus property holds if and only if there exists some $t_0 \in T$ such that the matrix A^{t_0} contains at least one strictly positive column [1].

Uniform Bounded Confidence Model. This model was introduced by Hegselmann and Krause [4] in their 2002 work: an agent's change in opinion over time is obtained by averaging the opinions of those agents which are ε-close to that agent ($0 \le \varepsilon \le 1$). For a given agent x_i, the fixed parameter ε helps us collect all agents x_j whose opinions will be considered by x_i. The index set of all ε-neighbors of agent x_i is given by (using the notation from [4]):

$$I(i, \vec{x}) = \{1 \le j \le N \mid |x_i - x_j| \le \varepsilon\}. \tag{1}$$

We also introduce the set of ε-close opinions to x_i:

$$O(x_i) = \{x_j \mid |x_i - x_j| \le \varepsilon\}. \tag{2}$$

Formula (1) states that if the distance between agent x_i and x_j is less than or equal to ε then that opinion will be considered by agent x_i. For example, for the set $\vec{x}$ described below and an ε value of 0.3, agent x_1 will consider the following subset of opinions (2) and corresponding index set (1):

$$\vec{x} = [0.29, 0.113, 0.97, 0.84, 0.57, 0.64, 0.52, 0.059, 0.33, 0.72]$$

$$I(1, \vec{x}) = \{1, 2, 5, 7, 8, 9\} \quad \text{and} \quad O(0.29) = \{0.29, 0.113, 0.57, 0.52, 0.059, 0.33\}$$

Essentially, the value ε creates a range for opinion consideration and is the reason it is called the bounded confidence interval.

To understand the dynamics of this model, we show the averaging process used for updating the values of the profile $\vec{x}(t)$. Using discrete time $t \in T = \{0, 1, 2, 3, \ldots\}$ and using the index set definition (1) for a given ε, the averaging formula is given by (see also [4]):

$$x_i(t+1) = \left|I\big(i, x(t)\big)\right|^{-1} \sum_{j \in I(i,x(t))} x_j(t) \quad \text{for } t \in T \qquad (3)$$

where $\left|I\big(i, x(t)\big)\right|$ denotes the cardinality of the index set $I\big(i, x(t)\big)$.

In practice, this formula is quite simple to implement. For each x_i, the following time step value $x_i(t+1)$ is the average of all the opinions that are considered by the agent x_i.

Example 1. Consider the opinion profile of $\vec{x}(0) = [0.4, 0.5, 0.7, 0.8]^T$ and an ε value of 0.2 the next time step would produce:

$$\vec{x}(1) = \left[\frac{(0.4+0.5)}{2}, \frac{(0.4+0.5+0.7)}{3} \, \frac{(0.5+0.7+0.8)}{3}, \frac{(0.7+0.8)}{2}\right]^T$$

Notice that this is for a single time step. For the following time step, the process repeats with the updated values of $\vec{x}(1)$. Upon close observation, one will notice that the future iterates $\vec{x}(t)$ would begin to approach each other until "consensus" is reached (in this example); more specifically, after four steps, the opinion profile $\vec{x}(4) = [0.6, 0.6, 0.6, 0.6]^T$.

In the case that an agent's opinion is "too far" from other agents (i.e., the bounded confidence interval will never include that agent's opinion in the converging average), there is no hope for consensus and thus "fragmentation" takes place. For example, for the same opinion profile above $\vec{x}(0) = [0.4, 0.5, 0.7, 0.8]^T$ and $\varepsilon = 0.15$, one has $\vec{x}(1) = [0.45, 0.45, 0.75, 0.75]^T$ and this opinion profile remains unchanged, showing fragmentation: the first two agents are split from the last two.

Properties of Bounded Confidence [5]:

I. The dynamics does not change the order of opinion, i.e., $x_i(t) \leq x_j(t)$ for all $i \leq j$ implies that $x_i(t+1) \leq x_j(t+1)$ for all $i \leq j$.

II. If a split between agents occurs it will remain split forever.

III. Given an initial profile $\vec{x}(0)$, consensus is reached in finite time if and only if the opinion profile $\vec{x}(t)$ is an ε-profile at all times. (An ε-profile for an ordered opinion vector means one where any two adjacent entries have a distance less than or equal ε.)

IV. For $n = 2, 3, 4$ a consensus is approached if and only if the initial profile is an ε-profile.

Property IV does not remain true if $n \geq 5$. The example below shows a profile of seven agent that form a 0.2-profile, for which consensus is not reached.

Example 2. Consider the initial (ordered) opinion profile of seven agents:

$$\vec{x}(0) = [0.12, 0.29, 0.43, 0.61, 0.73, 0.84, 0.93]$$

Notice in Figure 1 that although the profile is an ε-profile (for $\varepsilon = 0.2$) the time evolution shows fragmentation after 5 steps. Alternatively, consensus happens when $\varepsilon = 0.3$.

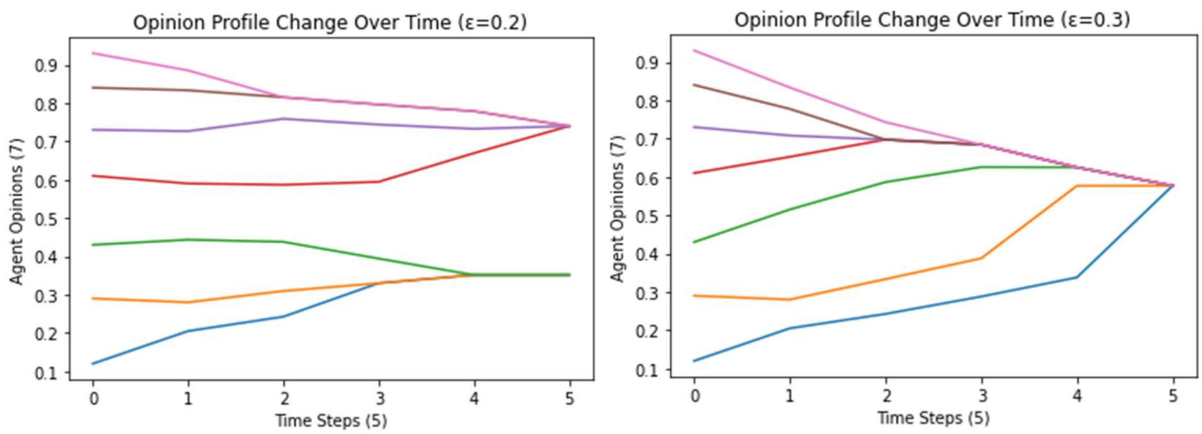


Figure 1 – 7 Agents with $\varepsilon = 0.2$ vs $\varepsilon = 0.3$

The principle of the third agent mentioned for the classical model (CM) says that consensus is reached if there exists a "third" agent that will bridge the gap between any two opposing opinions. A similar principle exists for the bounded confidence model: in this situation, a sufficient condition for consensus is that there exists t_0 such that $I\big(i, x(t)\big) \cap I\big(j, x(t)\big) \neq \emptyset$ for all i, j ∈ {1, … , N} and all $t \geq t_0$. It is much harder to check since it requires that a third connecting agent exists for all $t \geq t_0$.

Another sufficient condition obtained by Hegselmann and Krause [5, Theorem 5] is that consensus will be approached for a given initial opinion profile, provided that for an equidistant sequence {0, h, 2h, 3h, …} of time points for any two agents i and *j* there exists a third one *k* such that a chain of confidence leads from *i* to *k* as well as from *j* to *k* between *mh* and *(m+1)h*. In this case, the consensus is reached in finite time.

Chapter 2: Random Opinion Profile Simulations

2.1 Observing Random Profiles. The third agent principle is critical in determining the necessary conditions for consensus. To investigate these conditions further, we make use of numerical simulations to display an agent's change in opinion over time. By developing a python program for this, we can display graphs to show the behavior of consensus and fragmentation of a variety of opinion profiles (Appendix A). These simulations are inspired by the works done in Hegselmann and Krause's 2002 article for the one dimensional ε case [4]. For these simulations, a variety of profile sizes and ε values were conducted to determine if there was an identifiable pattern amongst these graphs. Such patterns include the conditions for convergence, fragmentation, and the number of groups formed.

Using the above formula (3), a random opinion profile was then updated at every time step t. In a separate vector, the updated value for every agent x_i at time t was recorded. The program continued to iterate until the distance between $\vec{x}(t+1)$ and $\vec{x}(t)$ were within a desired tolerance range using the following formula:

$$d\big(\vec{x}(t+1), \vec{x}(t)\big) = \|\vec{x}(t+1) - \vec{x}(t)\| \leq 0.5x10^{-8} \qquad (4)$$

In this case, the desired tolerance was assigned a value of $0.5x10^{-8}$. It was discovered that the tolerance had an influence on the time required to reach a stationary state by a few steps. Despite this, the tolerance did not affect consensus or fragmentation of any opinion profile. Since the objective was to investigate consensus or fragmentation conditions, tolerance was kept at an arbitrary static number. It should be noted that for any analysis of time to the stationary state mentioned later in the literature, the results may vary slightly based on this value.

After applying equation (4), the x_i agent vectors were plotted against the number of time steps required to reach the stationary state. In the first set of experiments, the number of agents was set to $N = 50$ and the value of ε was changed. Given the number of experiments that were conducted, a few interesting graphs were chosen as material for presentation to the reader. For example, Figure 2 illustrates the agent's change in opinion over time with $\varepsilon = 0.15$:

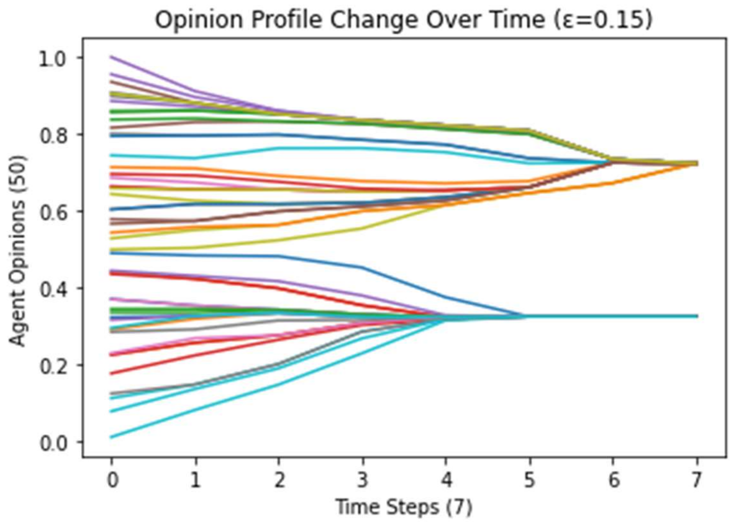


Figure 2 – 50 Agents & $\varepsilon = 0.15$

Notice that each agent begins to converge with neighboring agents that have similar opinions to their own. The value of ε, however, is too small for the entire opinion profile to reach consensus which leads to fragmentation into 2 groups.

Conversely, the larger the value of ε, the greater number of times consensus was reached. In Figure 3, the agent's change in opinion over time with $\varepsilon = 0.25$ shows a different outcome:

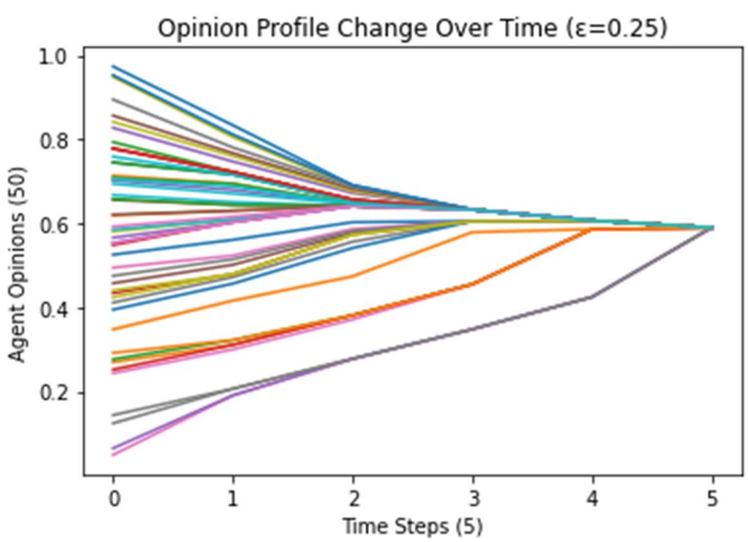


Figure 3 – 50 Agents & $\varepsilon = 0.25$

With this higher ε value, the 50 agents reach consensus in 5-time steps. It is important to mention that this graph is only a single instance of a random opinion profile and does guarantee convergence when $\varepsilon = 0.25$. The next set of experiments involved increasing the number of agents to 300 and attempting to find an ε value that would give an equal chance towards consensus or fragmentation.

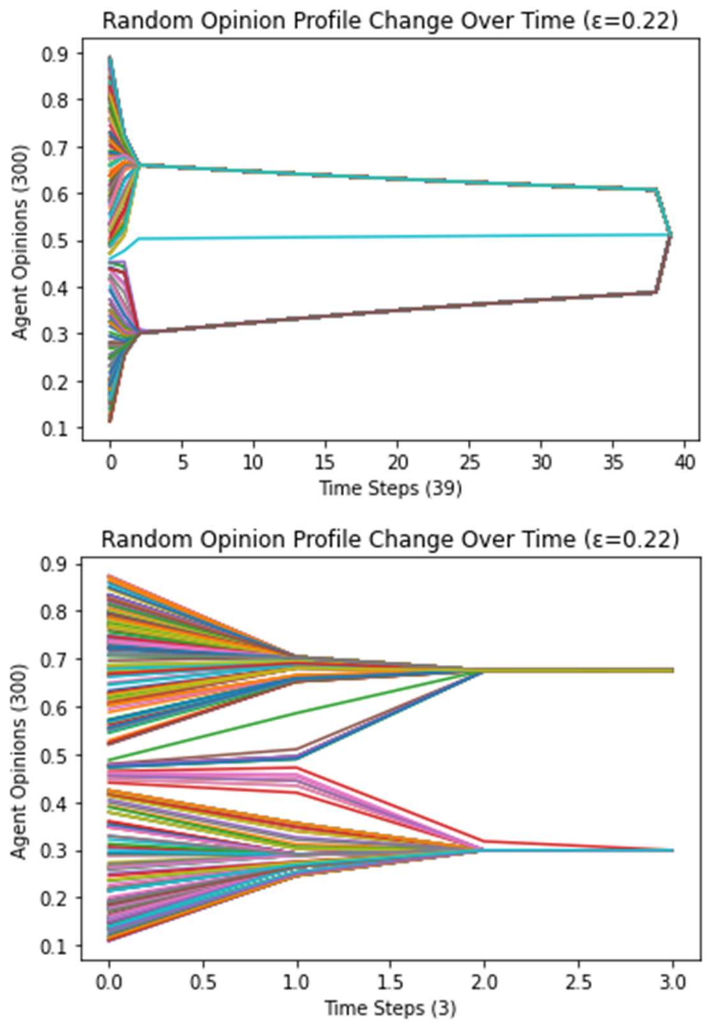


Figure 4 – 300 Agents & $\varepsilon = 0.22$

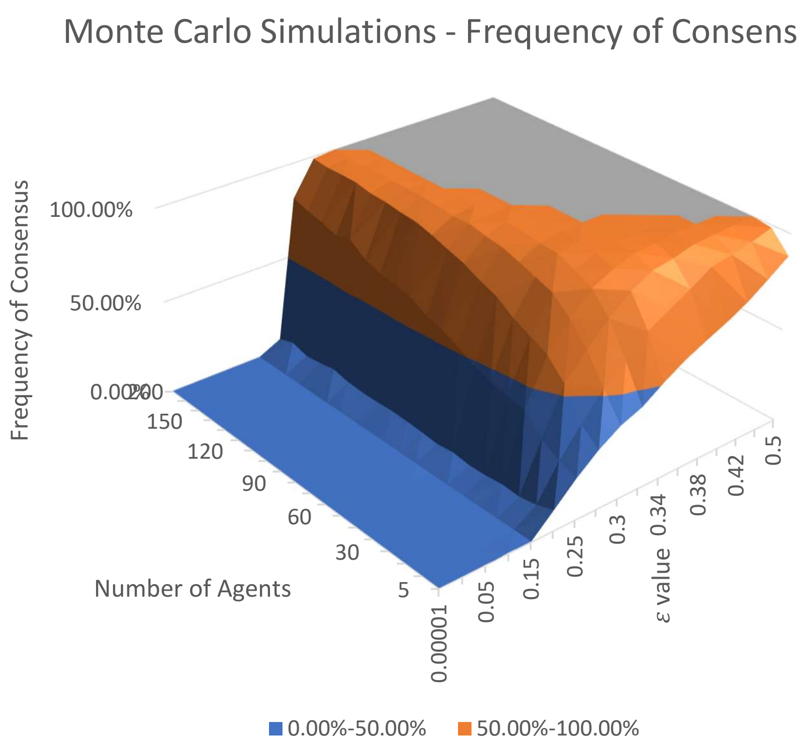
Figure 4 graphs were both generated with 300 agents and $\varepsilon = 0.22$. This is an example of ε where many different outcomes are possible. After a variety of experiments were conducted, it appeared that something notable was happening between the confidence interval of $0.2 \leq \varepsilon \leq 0.3$.

2.2 Monte Carlo Simulations. Unsurprisingly, the ε value had the largest effect on determining convergence or divergence of profiles in our experiments. What was interesting, however, was the effect of the number of agents and its relationship to defining a meaningful ε value for consensus. To investigate the frequency of consensus, we employ the use of Monte Carlo Simulations. An article published by the U.S National Library of Medicine succinctly defines the Monte Carlo method as a simulation "…[that] uses random sampling and statistical modeling to estimate mathematical functions and mimic the operations of complex systems" [3]. These numerical simulations allow for the prediction of an outcome from a complex system that has random variables present. It does this by simulating a random event repeatedly over a designated amount of time to provide the likelihood of an outcome. In this case, the outcome to be tested was the rate of consensus with a variety of profile sizes and ε values. This way, by using this series of repeated experiments, we can discover a more deterministic outcome from these random profiles.

To do this, another python script was developed to use this technique and test the number of times consensus was reached (Appendix A). Given a set consisting of the number of agents and a value of ε, the python program would conduct 10,000 simulations and record the number of times the profile fragmented and reached consensus. Once the frequency of consensus was determined for that set, a new set of agents and value of ε were simulated. This process was repeated until the number of agents was 200. Beyond this number of agents was too computationally expensive and was beyond the scope of this project's current capabilities.

By iterating these ε values and a given number of agents, one can begin to see the *frequency* of random opinion profiles reaching consensus (Appendix B). A 3D graphical representation was then created from the table of frequencies and produced the following plots below (see Figure 5).

From the graphs in Figure 5, one will notice there is an exponential growth from $\varepsilon = 0.2$ and $\varepsilon = 0.3$ in the frequency of consensus. Beyond $\varepsilon = 0.5$, the frequency of consensus remains the same. Similar simulations were performed by Drs. Hegselmann and Krause in their 2002 article where they observed the same rapid increase in consensus. Understanding this special interval of consensus below the $\varepsilon = 0.3$ threshold is the most pertinent part of our research. From the graphs in Figure 5, it would also seem that adding more agents reduces the ε requirements to reach consensus. More research will need to be conducted to uncover if this is strictly limited to the number of agents and the associated computational power needed.



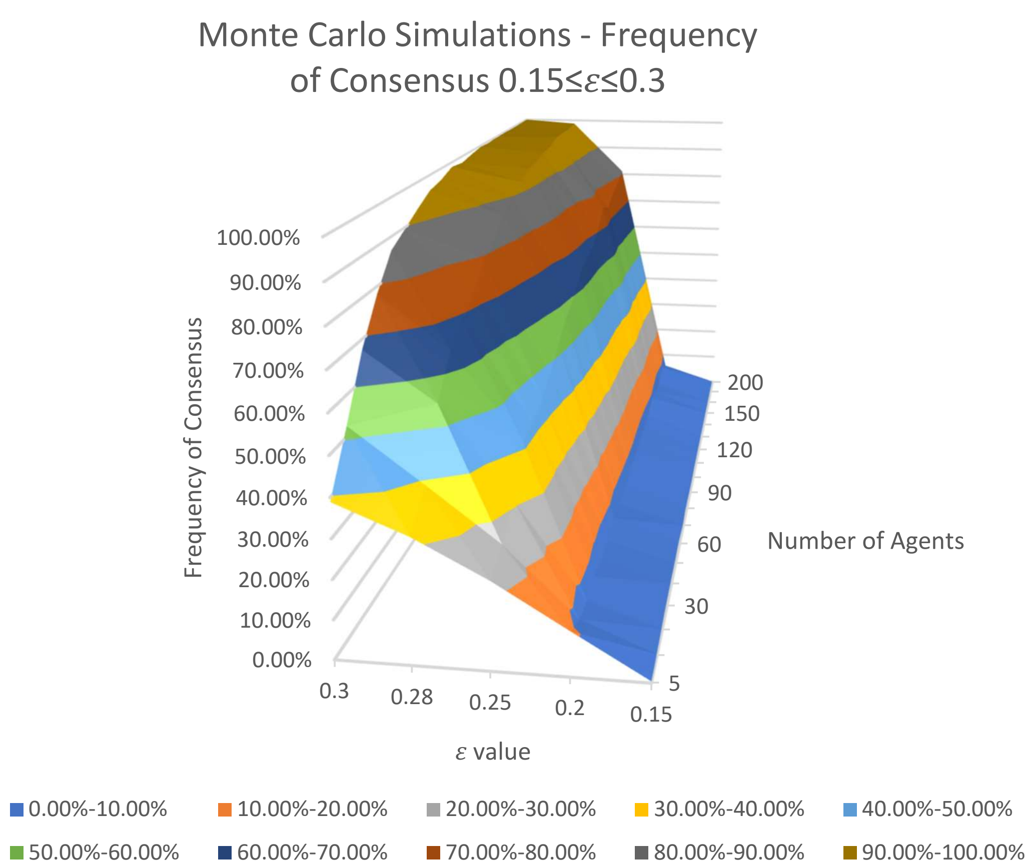


Figure 5 – Monte Carlo Frequency of Consensus Simulations

The following Table 1 shows a portion of the plotted points in Figure 5. It is clear to see that the frequency of consensus is almost 0% when $\varepsilon = 0.15$. Even when $\varepsilon = 0.2$, the frequency of consensus is less than 10%. Clearly, some phenomenon is causing this interval of $0.2 \leq \varepsilon \leq 0.3$ to have a dramatic increase in consensus.

Since much of the math behind random systems is not well understood, investigating this confidence interval $0.2 \leq \varepsilon \leq 0.3$ for the *random* case is limited to Numerical Simulations. To help us understand this behavior further, we introduce order to this random system by creating evenly spaced opinion profiles. This same subsequent approach was also taken by Drs. Hegselmann and Krause. In their 2002 article, the authors refer to this evenly distributed variety of opinions as "plurality" which we now seek to expand upon [4].

Monte Carlo Simulation – Consensus Frequency (10,000 Trials)						
# of agents	**0.15**	**0.2**	**0.25**	**0.28**	**0.3**	Ɛ Value
5	0.41%	11.20%	22.20%	31.60%	38.90%	
10	0.30%	7.80%	25.40%	39.20%	52.60%	
20	0.20%	6.20%	31.40%	54.80%	67.40%	
30	0.10%	6.70%	41.30%	64.30%	78.00%	
40	0.10%	5.60%	42.30%	68.90%	86.40%	
50	0.10%	5.30%	44.70%	73.20%	89.50%	
60	0.00%	6.40%	47.90%	79.40%	92.70%	
70	0.20%	5.30%	52.50%	82.70%	95.60%	
80	0.00%	6.10%	56.80%	86.90%	96.70%	
90	0.10%	6.70%	59.20%	89.60%	98.60%	
100	0.10%	7.00%	64.10%	92.50%	98.00%	
110	0.00%	6.50%	66.00%	94.80%	98.90%	
120	0.00%	6.40%	67.00%	95.40%	99.60%	
130	0.00%	7.00%	70.10%	96.50%	99.40%	
140	0.00%	8.80%	76.20%	96.80%	99.80%	
150	0.00%	7.30%	73.90%	98.40%	99.90%	
160	0.00%	6.10%	76.80%	98.40%	99.70%	
175	0.00%	8.60%	78.60%	98.10%	99.90%	
200	0.00%	5.70%	81.20%	98.80%	100.00%	

Table 1 – Consensus Frequency (10,000 Trials)

Chapter 3: Evenly Spaced Profile Analysis

3.1 General Properties of Evenly Spaced Profiles By investigating evenly spaced profiles, we believe it will provide insight into why this range is sufficient for consensus of randomly spaced profiles. If it could be shown, for evenly spaced profiles, that in this interval there is a significant increase in consensus without the randomness property, then it is possible to establish a relationship with ε in random profiles. To do this, instead of the opinion profiles being randomly generated, we seek to evenly split a profile of size N from 0 to 1.

Given this, we know that according to property (I) of bounded confidence [5], the dynamics does not change the order of opinions. For this reason, for the remainder of this literature, we will always order our opinion profile agent values from smallest to largest. An evenly spaced initial opinion profile of size N can then be described as follows:

$$\vec{x}(0) = \left[0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1\right] \quad \text{where} \quad n = N - 1$$

This situation is a particular case of symmetrical profiles, since $x_{N+1-i}(0) = 1 - x_i(0)$ for $i = 1,2,\dots,N$. The next proposition shows that the symmetry property is maintained for all iterates $\vec{x}(t)$.

Proposition 1. *For any ε value, if $\vec{x}(0)$ is a symmetric initial ordered profile of N agents, then $\vec{x}(t)$ is symmetric for all $t > 0$.*

Proof. The proof is by mathematical induction. The base case $\vec{x}(0) = \big(x_1(0),\ x_2(0), \dots, x_N(0)\big)$ is symmetric by design: $x_{N+1-i}(0) = 1 - x_i(0)$ for $i = 1,2,\dots,N$. (Notice that when N is odd, then the middle agent has an opinion value of 0.5 and this entry is symmetric with itself.) Assuming that the symmetrical property is true for $\vec{x}(t)$, we show that symmetry is true for $\vec{x}(t+1)$.

Using (2), suppose that the ε-close opinions to $x_i(t)$ are

$$O(x_i(t)) = \left\{ x_j(t) : \ \left|x_i(t) - x_j(t)\right| \le \varepsilon \right\} = \{x_{i-l}(t), x_{i-l+1}(t), \dots, x_i(t),\ x_{i+1}(t), \dots, x_{i+r}(t)\}$$

where $x_{i-l}(t)$ is the left-most ε-neighbor of $x_i(t)$ and $x_{i+r}(t)$ is its right-most ε-neighbor. Notice that $x_j(t)$ is ε-close to $x_i(t)$ if and only if $(1 - x_j(t))$ is ε-close to $(1 - x_i(t))$ which is equivalent to saying that $x_{N+1-j}(t)$ is ε-close to $x_{N+1-i}(t)$. In other words, the list of ε-neighbors of $x_{N+1-j}(t)$ coincides with the symmetrical entries of the ε-neighbors of $x_i(t)$.

$$O\big(x_{N+1-i}(t)\big) = \big\{x_{N+1-(i+r)}(t), x_{N+1-(i+r)+1}(t), \dots, x_{N+1-i}(t), x_{N+1-i+1}(t), \dots, x_{N+1-(i-l)}(t)\big\}$$
$$= \{1 - x_{i+r}(t), 1 - x_{i+r-1}(t), \dots, 1 - x_i(t), 1 - x_{i-1}(t), \dots, 1 - x_{i-l}(t)\}$$

One can also write that $O\big(x_{N+1-i}(t)\big) = 1 - O\big(x_i(t)\big)$, so by averaging the entries in each set we obtain

$$x_{N+1-i}(t+1) = 1 - x_i(t)$$

This shows that the symmetrical property is true for $\vec{x}(t+1)$ and, by mathematical induction, for all values $t \geq 0$. ■

Proposition 2. If an initial symmetric profile in $[0,1]^N$ reaches consensus, then the consensus value is 0.5.

Proof. We proved in Proposition 1, that if $\vec{x}(0)$ is symmetric, then $\vec{x}(t)$ is symmetric for all $t > 0$. If consensus is reached, meaning that $\lim_{t\to\infty} \vec{x}(t) = (c, c, \dots, c)$, then the limiting profile will also be symmetric, hence $c = 0.5$.

We now return to analyzing an evenly spaced initial opinion profile of size N:

$$\vec{x}(0) = \left[0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1\right] \quad \text{where} \quad n = N - 1.$$

In the situation where the bounded confidence value $\varepsilon < \frac{1}{n}$ it is clear that an agent x_i will never consider another neighbor x_j's opinion and the profile $\vec{x}(0)$ will remain stationary. What about the case when $\varepsilon = \frac{1}{n}$, exactly the size of the gaps between adjacent agents? From first glance, ε may appear to be sufficient to create consensus in this case as all the values are exactly $\frac{1}{n}$ distance away from one another, so $\vec{x}(0)$ is what in Chapter 1 was called an $\varepsilon = \frac{1}{n}$ - chain. This, however, is not the case.

Theorem 3. *For an opinion profile with N agents in $[0,1]$ that are ordered and evenly spaced, and $\varepsilon = \frac{1}{n}$ $(n = N-1)$, the opinion profile will reach fragmentation at $t = 5$.*

Proof. At $t = 1$, the opinion profile applying formula (3) would result in:

$$\vec{x}(1) = \left[\frac{0+\frac{1}{n}}{2}, \frac{0+\frac{1}{n}+\frac{2}{n}}{3}, \frac{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}}{3}, \frac{\frac{2}{n}+\frac{3}{n}+\frac{4}{n}}{3}, \dots\right] = \left[\left(\frac{1}{2n}\right), \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots\right]$$

At $t = 2$ the opinion profile applying formula (3) would result in:

$$\vec{x}(2) = \left[\frac{\frac{1}{2n}+\frac{1}{n}}{2}, \frac{\frac{1}{2n}+\frac{1}{n}+\frac{2}{n}}{3}, \frac{\frac{1}{n}+\frac{2}{n}+\frac{3}{n}}{3}, \frac{\frac{2}{n}+\frac{3}{n}+\frac{4}{n}}{3} \dots\right] = \left[\left(\frac{3}{4n}\right), \left(\frac{7}{6n}\right), \frac{2}{n}, \frac{3}{n} \dots\right]$$

At $t = 3$ the opinion profile applying formula (3) would result in:

$$\vec{x}(3) = \left[\frac{\frac{3}{4n}+\frac{7}{6n}}{2}, \frac{\frac{3}{4n}+\frac{7}{6n}+\frac{2}{n}}{3}, \frac{\frac{7}{6n}+\frac{2}{n}+\frac{3}{n}}{3}, \frac{\frac{2}{n}+\frac{3}{n}+\frac{4}{n}}{3}, \dots\right] = \left[\left(\frac{23}{24n}\right), \left(\frac{47}{36n}\right), \left(\frac{37}{18n}\right), \frac{3}{n}, \dots\right]$$

At $t = 4$ the opinion profile applying formula (3) would result in:

$$\vec{x}(4) = \left[\frac{\frac{23}{24n}+\frac{47}{36n}}{2}, \frac{\frac{23}{24n}+\frac{47}{36n}+\frac{37}{18n}}{3}, \frac{\frac{47}{36n}+\frac{37}{18n}+\frac{3}{n}}{3}, \frac{\frac{37}{18n}+\frac{3}{n}+\frac{4}{n}}{3}, \frac{\frac{3}{n}+\frac{4}{n}+\frac{5}{n}}{3}, \dots\right]$$
$$= \left[\left(\frac{163}{144n}\right), \left(\frac{311}{216n}\right), \left(\frac{229}{108n}\right), \left(\frac{163}{54n}\right), \frac{4}{n}, \dots\right]$$

Notice the gap between the first agent and the third agent which is $\frac{229}{108n} - \frac{163}{144n} = \frac{427}{432n} < \frac{1}{n}$. This shows that $\frac{229}{108n}$ is a sufficiently close neighbor to the first agent, hence $\frac{163}{144n}$ and $\frac{229}{108n}$ will be included in each other's average in the next time step.

At $t = 5$ the opinion profile applying formula (3) would result in:

$$\vec{x}(5) = \left[\frac{\frac{163}{144n} + \frac{311}{216n} + \frac{229}{108n}}{3}, \frac{\frac{163}{144n} + \frac{311}{216n} + \frac{229}{108n}}{3}, \frac{\frac{163}{144n} + \frac{311}{216n} + \frac{229}{108n} + \frac{163}{54n}}{4}, \frac{\frac{229}{108n} + \frac{163}{54n} + \frac{4}{n}}{3}, \dots\right]$$

$$= \left[\left(\frac{2027}{1296n}\right), \left(\frac{2027}{1296n}\right), \left(\frac{3331}{1728n}\right), \left(\frac{329}{108n}\right), \left(\frac{649}{162n}\right), \frac{5}{n}, \dots\right]$$

By examining the distance between agent x_3 and x_4 we observe the following:

$$\frac{329}{108n} - \frac{3331}{1728n} = \frac{1933}{1728n} > \frac{1}{n}$$

Thus, the agents x_3 and x_4 are no longer sufficiently close to be included in each other's averages. Since this is ordered from smallest to largest, it is clear to see that any further averages formed by this split will never converge again resulting in fragmentation of the opinion profile. This is further reinforced by property (II) of bounded confidence referenced before [5]. ■

We now investigate the case when an evenly spaced opinion profile has an *odd* number of agents.

3.2 Results and Simulations for Odd Numbered Profiles. Our first result shows that, although not optimal, $\varepsilon = 0.35$ is sufficient to guarantee convergence for all evenly spaced profiles with odd number of agents $N \geq 5$.

Theorem 4. *For all evenly spaced odd numbered profiles of size* $N \geq 5$ *and* $\varepsilon = 0.35$ *consensus is guaranteed in finite time based on the opinion profile at* $t = 2$. *Moreover, if* $N \geq 20$, *the opinion profile at* $t = 1$ *shows that consensus will be reached in finite time.*

Proof. Let us start by introducing *integer* $\ell = \lfloor \varepsilon n \rfloor$. Notice that ℓ is the maximum number of ε -neighbors that an agent in the initial profile will consider on each side. We write

$$\vec{x}(0) = \left[0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{\ell}{n}, \frac{\ell+1}{n}, \dots, 0.5, \dots, \left(1 - \frac{\ell+1}{n}\right), \left(1 - \frac{\ell}{n}\right), \dots, \left(1 - \frac{2}{n}\right), \left(1 - \frac{1}{n}\right), 1\right]$$

We show that the middle agent (with value) 0.5 is the third agent that exists as a bridge between the first and the last agent. Since we have proved the symmetrical property for every $t \geq 0$, we only need to calculate the opinion of the first agent to examine this bridge. More precisely, we want to show that $x_1(t = 2) \geq 0.15$.

We apply formulas (2) and (3) to update the initial profile $\vec{x}(0)$ resulting in $\vec{x}(1)$ as follows:

$O(0) = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{\ell}{n} \right\}$ so $x_1(t = 1) = \frac{\ell}{2n}$

$O\left(\frac{1}{n}\right) = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{\ell+1}{n} \right\}$ so $x_2(t = 1) = \frac{\ell+1}{2n}$

$O\left(\frac{2}{n}\right) = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{\ell+2}{n} \right\}$ so $x_3(t = 1) = \frac{\ell+2}{2n}$

….

$O\left(\frac{\ell}{n}\right) = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{\ell}{n}, \frac{\ell+1}{n}, \dots \frac{2\ell}{n} \right\}$ so $x_{l+1}(t = 1) = \frac{\ell+\ell}{2n}$

The agents with opinions between $\frac{\ell+1}{n}$ and 0.5 have a full symmetric set of left and right neighbors, so their opinions at time $t = 1$ remain unchanged.

Notice that if $n \geq 20$, then

$$x_1(t = 1) = \frac{\lfloor \varepsilon n \rfloor}{2n} \geq \frac{\varepsilon n - 1}{2n} = \frac{\varepsilon}{2} - \frac{1}{2n} \geq 0.175 - \frac{1}{40} = 0.15$$

This shows that if $n \geq 20$, then the middle (stationary) agent of value 0.5 is $\varepsilon = 0.35$-connected to the first entry $x_1(1)$ (and, by symmetry, to the last entry $x_N(1)$). Any future iterates $\vec{x}(t)$ will have the first entry $x_1(t) \geq x_1(1) \geq 0.15$, thus the middle agent is the $\varepsilon = 0.35$-connecting agent for all times $t \geq 1$. By the "third-agent property" described in Chapter 1 for bounded confidence profiles, we are guaranteed that consensus will be reached in finite amount of time.

We are left to check the cases when $n = 4, 6, 8, 10, 12, 14, 16, 18$ (recall that *N*=*n*+*1*). We compute explicitly the opinion profiles at time $t = 1$ and $x_1(t = 2)$ (if needed):

$n = 4$: $\vec{x}(1) = \left[\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}\right]$; $x_1(1) < 0.15$ but $x_1(2) = \frac{\frac{1}{8}+\frac{1}{4}}{2} = \frac{3}{16} > 0.15$

$n = 6$: $\vec{x}(1) = \left[\frac{2}{12}, \frac{3}{12}, \frac{4}{12}, \frac{1}{2}, \frac{8}{12}, \frac{9}{12}, \frac{10}{12}\right]$ and $x_1(1) \geq 0.15$

$n = 8$: $\vec{x}(1) = \left[\frac{2}{16}, \frac{3}{16}, \frac{4}{16}, \frac{3}{8}, \frac{1}{2}, \frac{11}{16}, \frac{12}{16}, \frac{13}{16}, \frac{14}{16}\right]$; $x_1(1) < 0.15$ but $x_1(2) = \frac{\frac{2}{16}+\frac{3}{16}+\frac{4}{16}+\frac{3}{8}}{4} = \frac{15}{64} > 0.15$

$n = 10$: $\vec{x}(1) = \left[\frac{3}{20}, \frac{4}{20}, \frac{5}{20}, \frac{6}{20}, \frac{4}{10}, \frac{1}{2}, \frac{6}{10}, \frac{14}{20}, \frac{15}{20}, \frac{16}{20}, \frac{17}{20}\right]$ and $x_1(1) \geq 0.15$

$n = 12$: $\vec{x}(1) = \left[\frac{3}{24}, \frac{4}{24}, \frac{5}{24}, \frac{6}{24}, \frac{4}{12}, \frac{5}{12}, \frac{1}{2}, \frac{7}{12}, \frac{8}{12}, \frac{18}{24}, \frac{19}{24}, \frac{20}{24}, \frac{21}{24}\right]$ and $x_1(1) \geq 0.15$

$n = 14$: $\vec{x}(1) = \left[\frac{4}{28}, \frac{5}{28}, \frac{6}{28}, \frac{7}{28}, \frac{8}{28}, \frac{5}{14}, \frac{6}{14}, \frac{1}{2}, \ldots\right]$; $x_1(1) < 0.15$

but $x_1(2) = \frac{\frac{4}{28}+\frac{5}{28}+\frac{6}{28}+\frac{7}{28}+\frac{8}{28}+\frac{5}{14}+\frac{6}{14}}{7} = \frac{52}{196} > 0.15$

$n = 16$: $\vec{x}(1) = \left[\frac{5}{32}, \frac{6}{32}, \frac{7}{32}, \frac{8}{32}, \frac{9}{32}, \frac{10}{32}, \frac{6}{16}, \frac{7}{16}, \frac{1}{2}, \frac{9}{16}, \frac{10}{16}, \frac{22}{32}, \frac{23}{32}, \frac{24}{32}, \frac{25}{32}, \frac{26}{32}, \frac{27}{32}\right]$ and $x_1(1) \geq 0.15$

$n = 18$: $\vec{x}(1) = \left[\frac{6}{36}, \frac{7}{36}, \frac{8}{36}, \frac{9}{36}, \frac{10}{36}, \frac{11}{36}, \frac{12}{36}, \frac{7}{18}, \frac{8}{18}, \frac{1}{2}, \frac{10}{18}, \frac{11}{18}, \frac{24}{36}, \frac{23}{36}, \ldots\right]$ and $x_1(1) \geq 0.15$

These additional cases show that the middle (stationary) agent of value 0.5 is $\varepsilon = 0.35$-connected to the first entry $x_1(t = 2)$ (and, by symmetry, to the last entry $x_N(t = 2)$) for all $n \geq 4$, which means $N = 5$ or more agents. Any future iterates $\vec{x}(t)$, will have the first entry $x_1(t) \geq x_1(2) \geq 0.15$, thus the middle agent is the $\varepsilon = 0.35$-connecting agent for all times $t \geq 2$. By the "third-agent property" described in Chapter 1 for bounded confidence profiles, we are guaranteed that consensus will be reached in finite time. ■

Through numerical simulations, we observe that an evenly spaced profile for an odd number of agents will reach consensus with a much smaller ε value than 0.35. We determined that <u>$\varepsilon = 0.2$</u> is sufficient for *odd* numbered profiles of size $N \geq 17$ to reach consensus.

In Figure 6, the number of agents simulated was $N = 27$, this graph reaches consensus after 18 time steps. Subsequently, we simulate an increase in the number of agents to 301 in Figure

7; we observe a dramatic increase in time steps required to reach the stationary state. Figure 7 takes 103 steps to reach consensus which suggests that an increase in N greatly affects the convergence time for this value of ε.

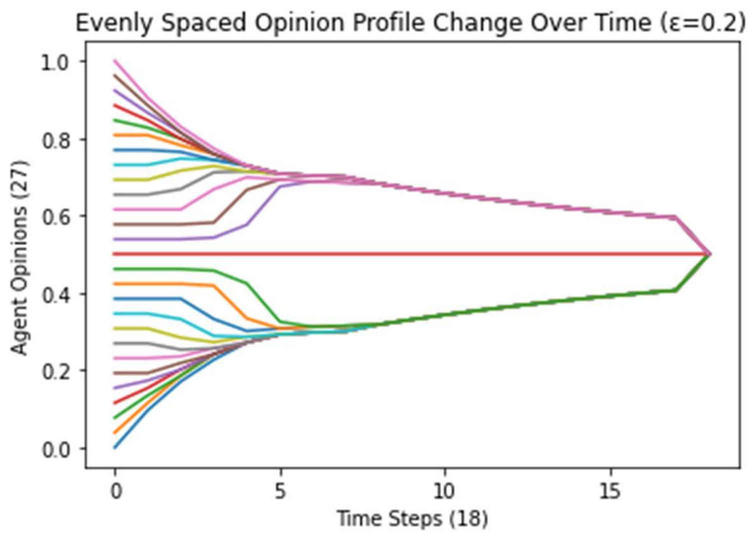


Figure 6 – Evenly spaced 27 Agents & $\varepsilon = 0.2$

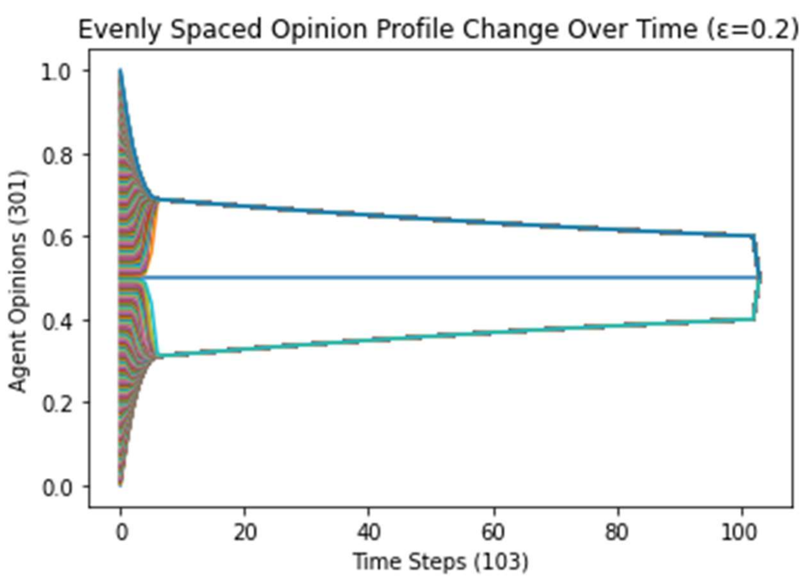


Figure 7 – Evenly spaced 301 Agents & $\varepsilon = 0.2$

A pattern that we observe frequently for an evenly spaced opinion profile of an odd numbered N is the formation of 3 groups to reach consensus. These three groups consist of the leftmost group with agent values equal, the center group all consisting of opinions equal to 0.5 and the group on the right whose agents also equal each other (and are symmetric to the leftmost group). Our findings suggest that at the end of time step 8, an analysis of the profile will determine if the agents' values within the left and right most groups are ε close to 0.5.

Similar to the proof in Theorem 4, identifying if the leftmost *agent* is ε close to 0.5 is sufficient in determining if the profile will reach consensus using the third agent principle. If it is, then by the third agent principle, consensus will be reached in finite time. (Theorem 3 shows that it is not enough for a series of agents to simply be an ε-profile.)

There are additional instances of this third agent principle creating consensus for values below the $\varepsilon = 0.2$ threshold. One such example is Figure 8 with $\varepsilon = 0.19$ and $N = 25$:

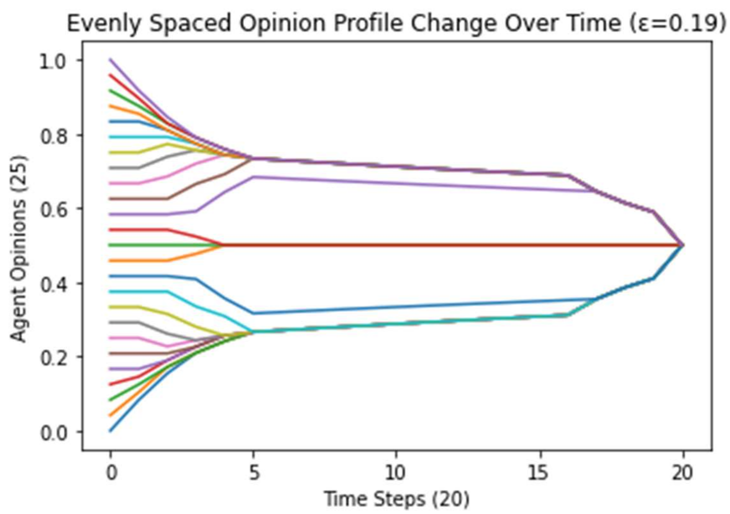


Figure 8 – Evenly spaced 25 Agents & $\varepsilon = 0.19$

These rarer cases will often have 5 groups associated with them and will still reach consensus. To determine how often consensus would occur below 0.2, we tested every odd number between 3 and 2003 for $\varepsilon = 0.19$ (1000 total). We discovered that 90.6% of these profiles fragmented and 9.4% reached consensus showing a dramatic decrease from $\varepsilon = 0.2$. Beyond 691 agents at $\varepsilon = 0.19$, no odd numbered profile up to 2003 was sufficient to reach consensus. This could indicate that $N = 691$ is the upper bound of agents needed for this special consensus or it could be a limitation in the quantity of numbers used. More research should be conducted to distinguish these possibilities.

For evenly spaced opinion profiles, $\varepsilon = 0.15$, is even more unlikely to generate consensus. Running the same test for odd numbers from 3 to 2003 with $\varepsilon = 0.15$ resulted in 96.6% fragmentation and 3.4% consensus. One such example of consensus at this ε value is as follows:

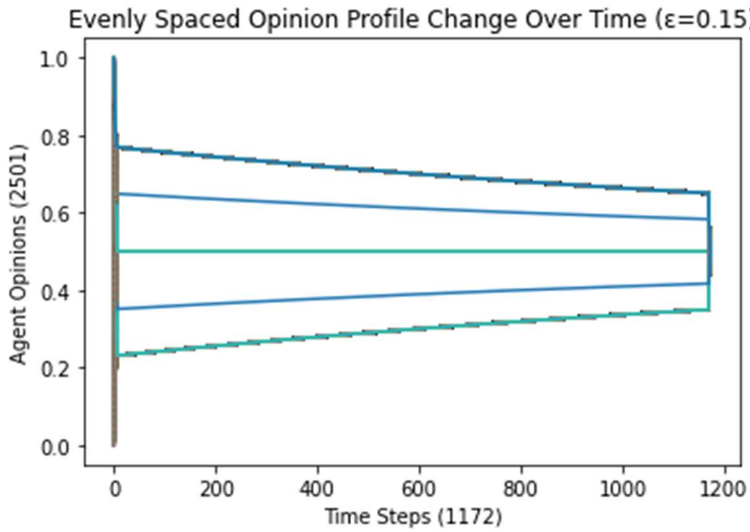


Figure 9 – Evenly spaced 2501 Agents & $\varepsilon = 0.15$

With $\varepsilon = 0.15$ and the number of agents equal to 2501, the simulation in Figure 9 is produced. In this special case, 5 groups were formed that were equidistant from each other. This profile took 1172-time steps for it to converge and was the largest increase in time steps to the stationary state from all our conducted experiments. What was particularly interesting about this simulation is that again at time step 8, there is a visible ε distance between immediate neighbors that suggests the profile could reach consensus.

The example in Figure 8 raises questions about the existence of additional properties of these ε-chains. From our observed experiments, these rare equidistant profiles were the most computationally expensive to implement. By time step 8, however, we can observe that these profiles form 5 groups with 1 additional group than normal on either side of the center group. Through these observations, we notice that if these 5 groups are ε close to their nearest neighbors they will reach consensus (i.e., an ε-profile).

In Theorem 3, we prove that agents being ε close to their immediate neighbor is not enough to produce an ε-chain that guarantees consensus. Despite this, we have not been able to find an example where an equidistant profile that was within ε of every immediate neighbor at time step

8 has not reached consensus. We hope to formally prove the existence of additional properties of ε-chains and consensus of equidistant profiles in a future publication.

3.3 Results and Observations for Even Numbered Profiles

Similarly to Theorem 4 it seems plausible that the following is true: *for an evenly-spaced profile of even number of agents* $N \geq 4,\ \varepsilon = 0.4$ *will guarantee consensus at* $time\ t = 2$. The new challenge here is the absence of a middle (stationary) agent of opinion value 0.5 (at least for the first time steps). Currently, we do not have a complete proof of this statement.

Through numerical simulations, we observe that an evenly spaced profile with an *even* number of agents will reach consensus with a much smaller ε value than 0.4. Our analysis shows that the necessary confidence interval for an even size N to reach consensus is $\varepsilon = 0.25$ for $N \geq 26$. Clearly, this is higher than that of the odd numbered profile with $\varepsilon = 0.2$.

To study this change, we investigate this value for guaranteed consensus with simulations using a variety of profile sizes and ε values. In Figure 10, a simulation consisting of 26 evenly spaced agents and $\varepsilon = 0.25$ are conducted. Using $\ell = \lfloor \varepsilon n \rfloor$, we know that $\ell = \lfloor 0.25 * 26 \rfloor = 6$ implying that the gap size of this simulation is 6 times greater than the gap size of $\frac{1}{n}$. Given these conditions, the evenly spaced opinion profile reached consensus within 7-time steps. We again alter the number of agents to determine its effect on the profile for this static ε. For Figure 11, the number of agents is increased to 300 agents with $\ell = 125$. Despite the significant increase in the number of agents captured, the profile still reaches consensus after 9-time steps. This suggests that the number of agents do not affect the speed of convergence greatly for this value of ε.

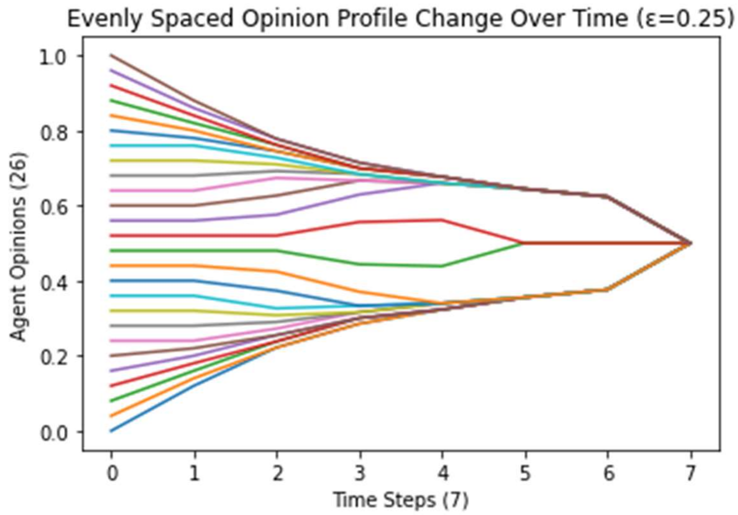


Figure 10 – Evenly spaced 26 Agents & $\varepsilon = 0.25$

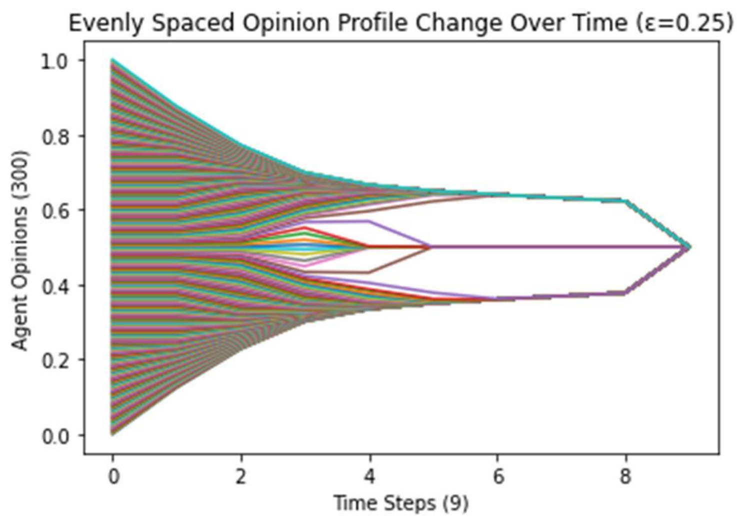


Figure 11 – Evenly spaced 300 Agents & $\varepsilon = 0.25$

Even numbered profiles are special in the sense that a particular ε value can form a bifurcation or what is referred to as polarization in Hegselmann and Krause's 2002 article [4]. This phenomenon is most abundantly seen in the interval $0.2 \leq \varepsilon < 0.25$. Since even numbered N profiles do not have any center values, the bifurcation formed creates a larger gap between the left and right most groups than odd numbered N profiles have. This explains the necessary increase in the ε value since the center connecting value 0.5 is not present in the initial profile.

As stated above, with even opinion profiles of size N, we observe that $\varepsilon \geq 0.25$ is sufficient to guarantee consensus for profile sizes of $N \geq 26$. In many cases, this value of ε produces several center values after 6-time steps that allow the profile to reach consensus more easily. We also observe that in the case where no center values are created in even numbered profiles of size N, consensus can only be reached when the distance between these polarized groups is ε close to each other.

The key observation of time step 8 analysis also applies to examine an evenly spaced opinion profile with an even N. In the case where $\varepsilon \geq 0.25$, most of the profiles observed reached convergence by time step 8. This further emphasizes the potential importance of this time step when analyzing opinion profiles.

Like in the odd numbered N case, there are a variety of conditions that lead to consensus for $\varepsilon < 0.25$. One such example is Figure 12 with $N = 816$ and $\varepsilon = 0.21$. In this simulation, consensus is reached in 64 times steps with the familiar 3 groups. In Figure 13, however, with a much higher $\varepsilon = 0.24$ and $N = 42$ we observe polarization.

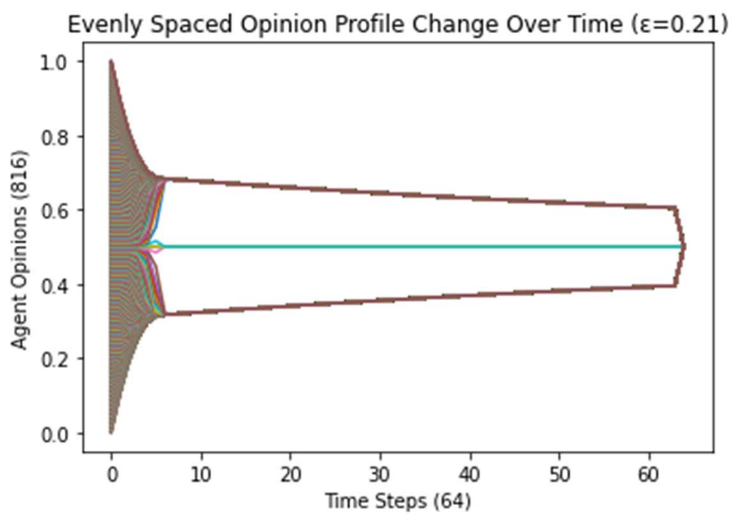


Figure 12 – Evenly spaced 816 Agents & $\varepsilon = 0.21$

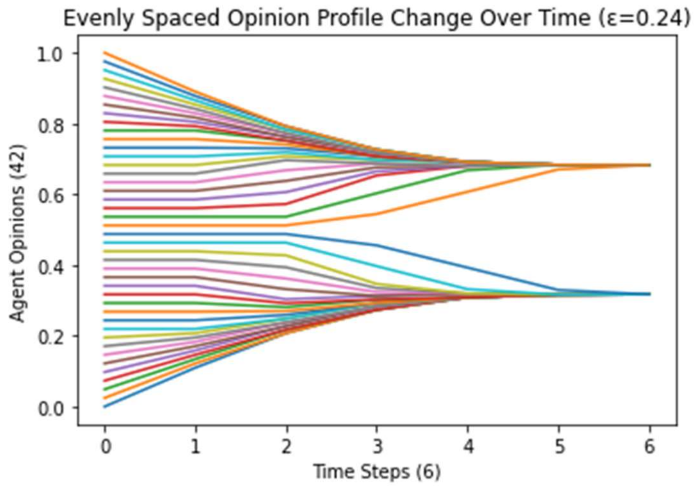


Figure 13 – Evenly spaced 42 Agents & $\varepsilon = 0.24$

To explore $\varepsilon < 0.25$ for profiles with an even number N, we use the same method as before and simulate all even numbers between 10-2002 (996 total) to get the percentage of profiles reaching consensus. Surprisingly, $\varepsilon = 0.24$ resulted in 98.4% of profiles reaching consensus and only 1.6% of profiles fragmenting. When $\varepsilon = 0.2$ we observed that 33.1% of profiles still reached consensus and 66.9% of profiles fragmented. Furthermore, when $\varepsilon = 0.19$ we observed that 11.8% of profiles still reached consensus and 88.2% of profiles fragmented. Lastly, when $\varepsilon = 0.15$ we observed that 3.7% of profiles still reached consensus and 96.3% of profiles fragmented. In Figure 14, we again observe the rarer case of an equidistant profile that reaches consensus. Again, at time step 8, we observe the ε distant neighbors that suggests the profile could converge.

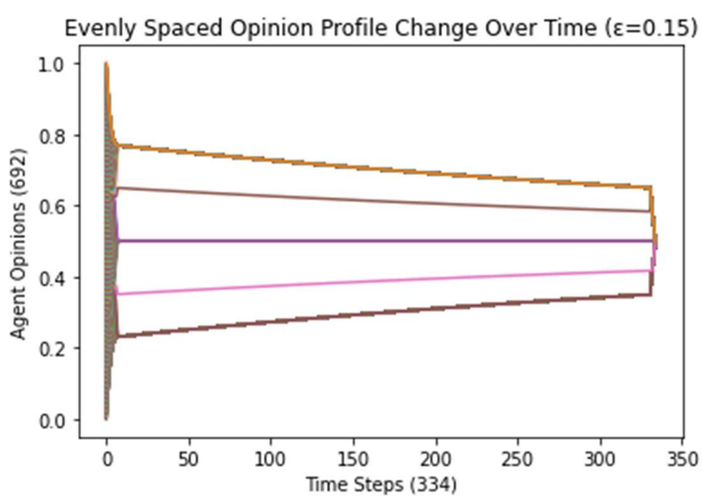


Figure 14 – Evenly spaced 692 Agents & $\varepsilon = 0.15$

3.4 Fragmentation of Evenly Spaced Profiles

Our analysis has shown general properties and conditions for consensus of these evenly spaced opinion profiles. We now seek to explore the conditions that lead to fragmentation and study these opinion profiles of an odd and even numbered N with a variety of conditions. For example, Figure 15 (a) and (b) simulates an opinion profile size of 25 and 26 agents, respectively with $\varepsilon = 0.15$ $(\ell = 3)$. After 7-time steps, both profiles fragment into 3 groups. In Figure 16 (a) and (b), the number of agents is increased to investigate if the number of group formations change for this fixed ε. The number of agents is subsequently increased to 201 and 200 with $\varepsilon = 0.15$ $(\ell = 30)$. This opinion profile is nearly identical to Figure 15, reaching the stationary state in 7 steps with 3 groups. We observe that for this value of ε, the number of agents does not vary the size of the groups formed.

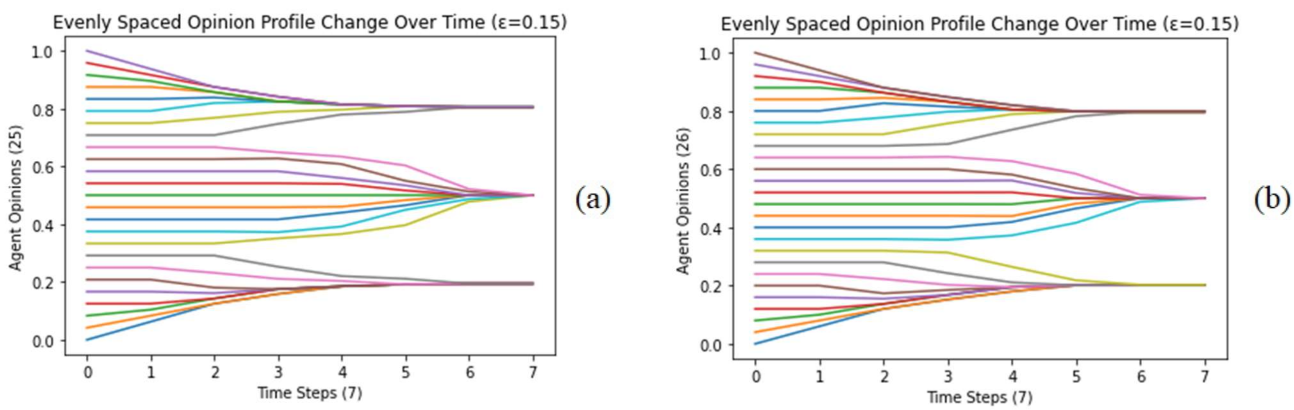


Figure 15 – Evenly spaced 25 Agents (a) & 26 agents (b) with $\varepsilon = 0.15$

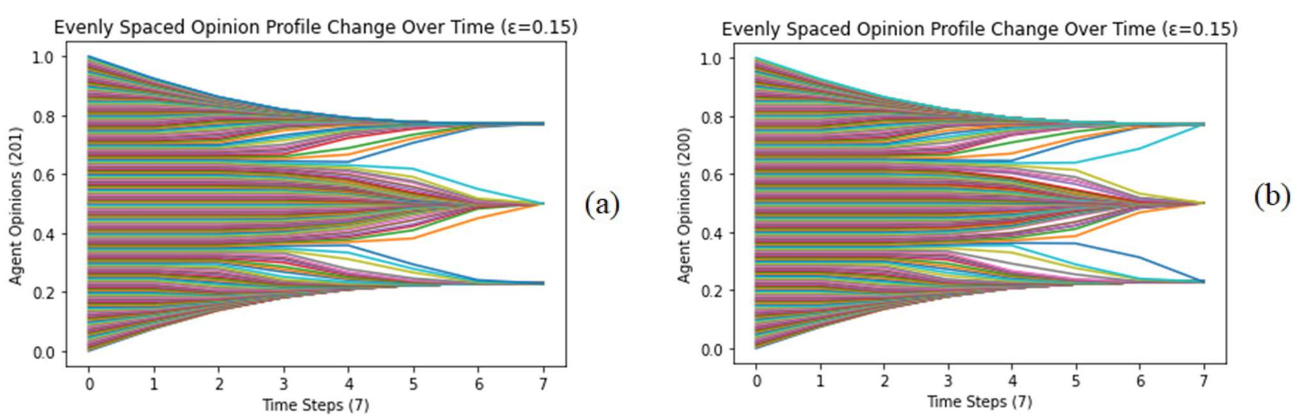


Figure 16 – Evenly spaced 201 Agents (a) & 200 Agents (b) with $\varepsilon = 0.15$

Lowering the confidence value to $\varepsilon = 0.1$ demonstrates some additional differences between evenly spaced profiles of odd and even numbered N. In Figure 17 (a) and (b), the evenly spaced profile size was set to 101 and 100 agents respectively with a corresponding multiple $\varepsilon = 0.1$ ($\ell = 100$). These simulations differ in the time steps required to reach the stationary state. In Figure 17 (a) it takes the profile 21 steps to reach the stationary state with three groups being formed. In Figure 17 (b), however, it takes the profile 14 steps to reach the stationary state with 5 groups being formed.

For Figure 18 (a) and (b) the number of agents increases to 1001 and 1000 respectively with a corresponding value of $\varepsilon = 0.1$ ($\ell = 150$). It becomes clear that increasing the size of this multiple does not affect consensus or fragmentation, rather it affects the number of groups being formed at the stationary state. In this case, it only takes 13 steps to reach this state for both profiles. In 18 (a) 5 groups are formed by these conditions while in 18 (b), 4 groups are formed. This is likely due to the even profile not having a 0.5 value at $t = 0$.

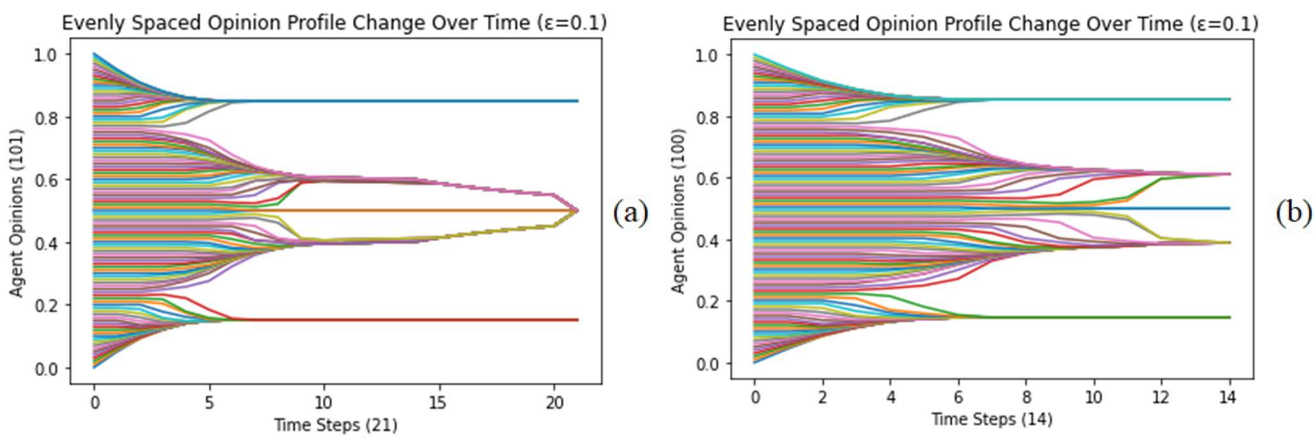


Figure 17 – Evenly spaced 101 Agents (a) & 100 Agents (b) with $\varepsilon = 0.1$

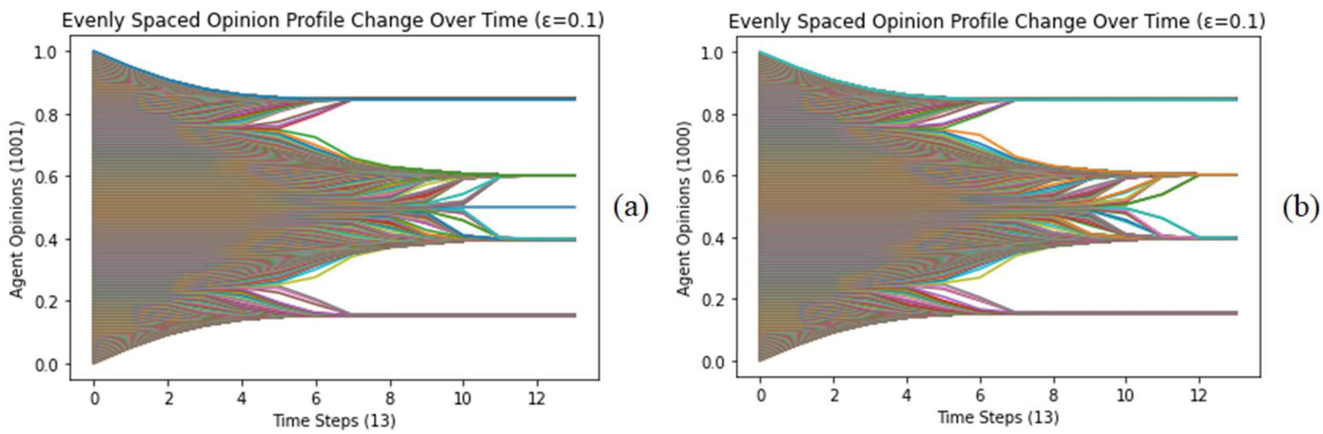


Figure 18 – Evenly spaced 1001 Agents (a) & 1000 Agents (b) with $\varepsilon = 0.1$

Lastly, we look to investigate $\varepsilon = 0.05$ and differentiate the number of groups formed by this change. Figure 19 (a) and (b) is a simulation conducted with 51 and 50 agents respectively with a corresponding multiple $\varepsilon = 0.05$ ($\ell = 2$). Figure 19 (a) profile fragments, reaching the stationary state at 29 steps with 7 groups formed. Figure 19 (b) profile fragments faster, reaching the stationary state at 17 steps with 9 groups formed.

Figure 20 (a) and (b) simulates an increase in the number of agents to 501 and 500 respectively with $\varepsilon = 0.05$ ($\ell = 25$). This simulation is one in which opinion profiles of both even and odd numbered N behave significantly different. In Figure 20 (a) the profile reaches the stationary state in 18-time steps with 9 groups. In Figure 20 (b), however, it takes 92-time steps to reach the stationary state while forming 7 groups.

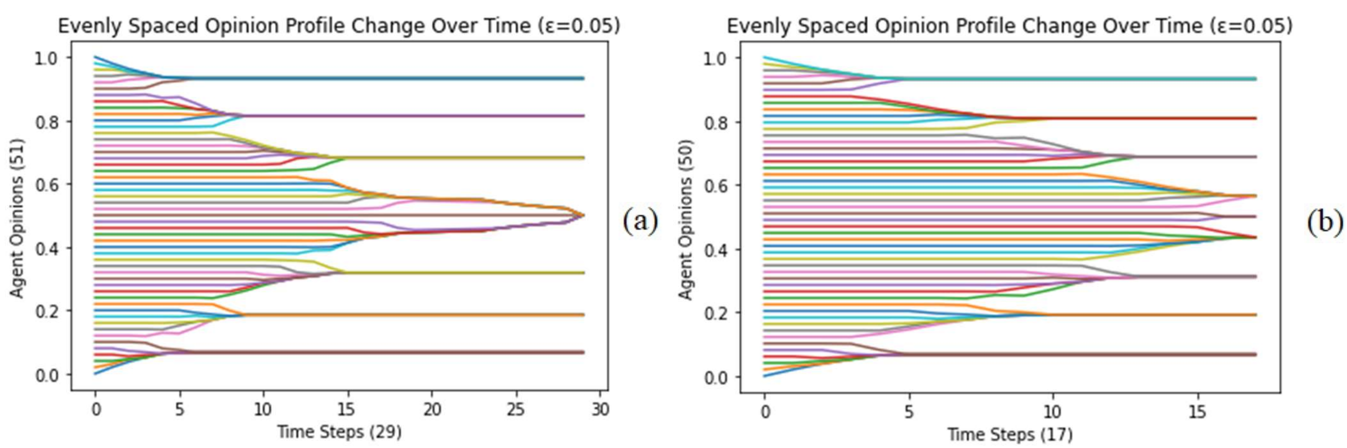


Figure 19 – Evenly spaced 51 Agents (a) & 50 Agents (b) with $\varepsilon = 0.05$

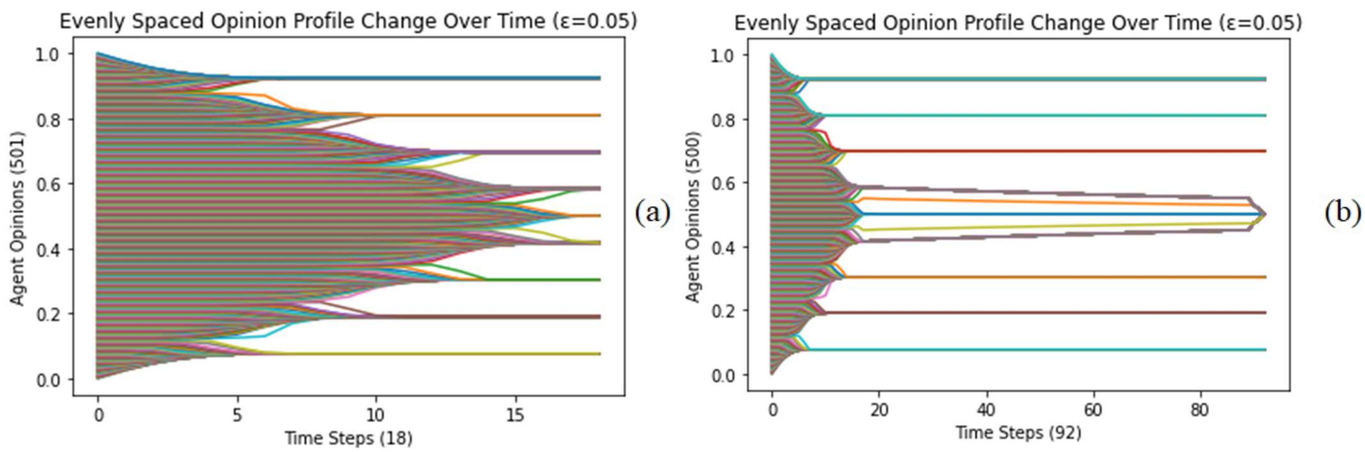


Figure 20 – Evenly spaced 501 Agents (a) & 500 Agents (b) with $\varepsilon = 0.05$

From these simulations, it is clear to see that when subgroups converge in a fragmented profile, the time to the stationary state increases significantly. It is also clear that decreasing ε increases the number of groups at the stationary state. This is an intuitive observation as more agents are captured in the iterating average, there is a greater number of convergences between agent opinions resulting in less groups formed. Conversely, if less agents are captured with a smaller ε value than more groups will form.

3.5 Additional Observations and Time to Consensus. Recall that in Theorem 3 we proved that for an evenly spaced initial profile

$$\vec{x}(0) = \left[0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-2}{n}, \frac{n-1}{n}, 1\right]$$

and $\varepsilon = \frac{1}{n}$ fragmentation happens after 5 time steps between agent 3 and 4. Numerical simulations show that groups of three equal entries are formed at this time step. Moreover, as ε increases the size of each group grows, thus decreasing the number of groups that form. The nature of the prevalent group size w that forms during these simulations is approximately given by

$$w \approx 2\ell + \left\lfloor \frac{\ell}{3} \right\rfloor \qquad (5)$$

where $\ell = \lfloor \varepsilon n \rfloor$ as described in theorem 4. After running numerous simulations, we conclude that the value of w is approximately equal to the size of the left and right most groups that form during step 8. This observation may also be the reason why we are able to analyze a profile at time step 8. If a profile fragments, this same group size (for the left and right most groups) is seen at the final time step. Lastly, if a profile reaches consensus, then these groups converge to 0.5 and the groups dissipate in the final time step.

We do notice, however, as ε increases, (5) loses accuracy which would suggest that the behavior of the group size is still not quite understood. By improving this formula, we may be able to develop insight on *when* a profile forms the 3 groups to consensus for non-equidistant profiles. This is one potential avenue in which to pursue the proof that time step 8 is sufficient for profile analysis.

Although the proof for step 8 remains elusive, we can investigate the time steps required to reach consensus. We begin by fixing $\varepsilon = 0.2$ to study the time to consensus for odd numbers. By increasing the number of agents and plotting the resulting time steps we produce Figure 21 (Appendix C). By visual inspection, Figure 21 shows the number of agents has a significant effect on the time to convergence.

What is notable is that, at 1801 agents, the profile begins to oscillate in time steps required. Since this graph was created by increments of 100, many number's time steps are not captured. We hypothesize that the profile will oscillate significantly more if all numbers and their corresponding time steps are recorded. We also believe that the amount of time steps required will increase linearly with the number of agents as indicated by Figure 21.

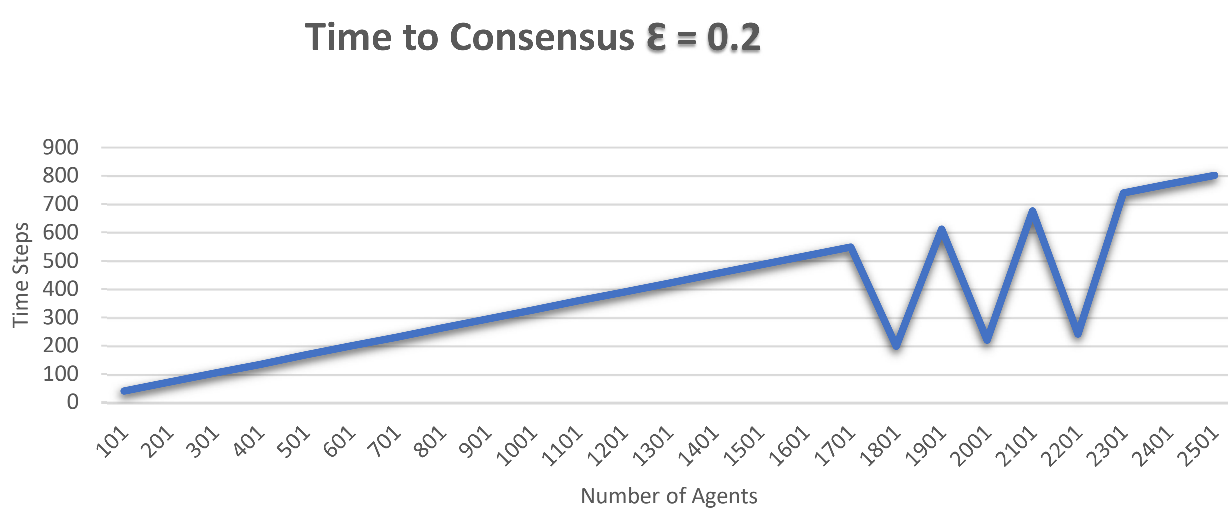


Figure 21 – Time to Consensus (odd number of agents) with fixed $\varepsilon = 0.2$

Lastly, we seek to investigate the consensus requirements for evenly spaced profiles of odd and even numbered N by fixing $\varepsilon = 0.25$. We again plot this against an increasing number of agents to observe the changes in time steps required (Appendix C). In Figure 22, it is clear to see that the number of agents has little effect on the number of time steps required to reach consensus for this value of ε. The longest convergence time was 10-time steps with an evenly spaced profile consisting of 200 agents. In Figure 23, the same behavior can be observed with evenly spaced profiles with an odd N. As the size of the profile increases the time steps required to reach the stationary state fixate at time step 8. In the odd numbered N case, since $\varepsilon = 0.2$ is already sufficient for consensus, it is reasonable to assume the time steps required for a larger ε would decrease the processing time. For the even numbered N case, in the interval $0.2 \leq \varepsilon \leq 0.25$, the frequency of consensus is still quite high. We hypothesize that once $\varepsilon = 0.25$, increasing profile sizes will continue to require no more than 8-time steps.

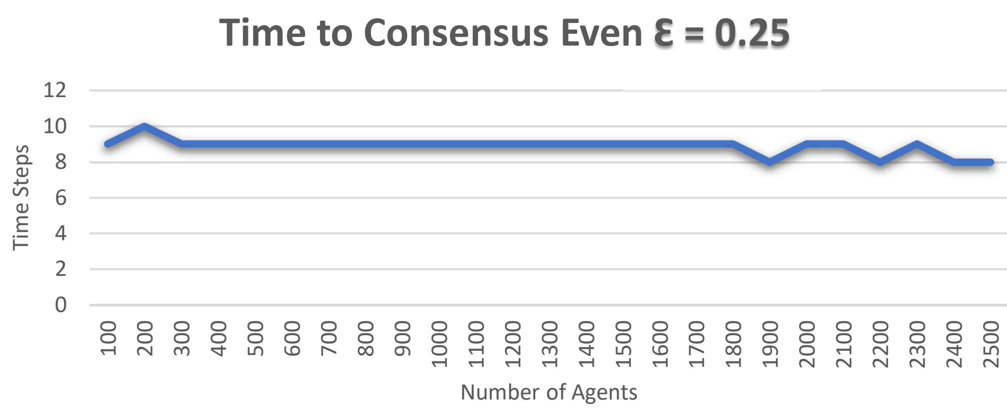


Figure 22 – Time to Consensus (even number of agents) with fixed $\varepsilon = 0.25$

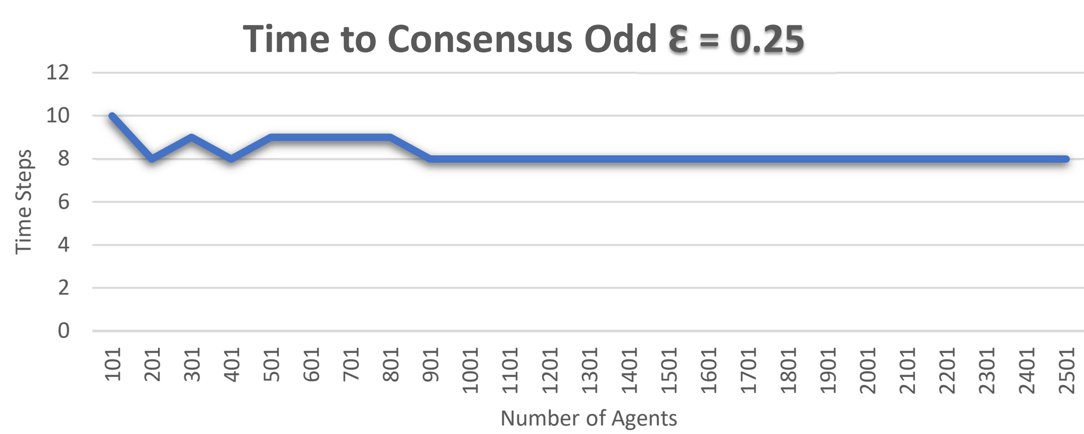


Figure 23 – Time to Consensus (odd number of agents) with fixed $\varepsilon = 0.25$

Conclusion

Using Monte Carlo Simulations, we observed the same result originally discovered in Drs. Hegselmann and Krause's 2002 paper [4]. Namely, a randomly spaced opinion profile has a dramatic increase in the interval $0.2 \leq \varepsilon \leq 0.3$. Exploring this phenomenon in the random case naturally led to an analysis of evenly spaced opinion profiles as a form of studying this complex system in a contained setting. From this, we sought to expand upon the work of plurality in the hopes of gaining new insights into these random systems.

Our work was able to prove that a gap size of $\frac{1}{n}$ $(n = N - 1)$ will guarantee fragmentation in 5-time steps. We were also able to prove that 0.35 guarantees consensus for $N \geq 5$ of an evenly spaced profile of an odd number of agents N. Our analysis also showed that lower values of ε are sufficient to reach consensus via numerical simulations. These simulations showed that 0.2 is sufficient for consensus for *odd* numbered evenly spaced profiles of size $N \geq 17$. Similarly, they showed 0.25 is sufficient for consensus for *even* numbered evenly spaced profiles of size $N \geq 26$. Most importantly, our research indicated that after 8-time steps, an evenly spaced opinion profile can be analyzed to determine consensus or fragmentation. These findings also suggested that observing an ε-profile at time step 8 may be enough to determine consensus for $t > t_0$.

Our experiments regarding time to consensus and rate of convergence also indicate that this bounded confidence algorithm is rather slow to converge, especially in the case of equidistant profile consensus. If the step 8 analysis can be formally proven to converge, this would greatly affect the processing time for this algorithm. Lastly, our analysis also identified numerous patterns between ε and the number of agents in a profile. This included the approximation of the left and right most group size formations, even and odd number relationships and the rate of convergence.

More research is needed to determine what relationship, if any, these confidence values 0.2 and 0.25 have to the randomly spaced opinion profiles. In a future literature, we hope to be able to explore this and formally prove that step 8 is sufficient for profile analysis. We also hope to prove $\varepsilon = 0.2$ and $\varepsilon = 0.25$ are sufficient for consensus for evenly spaced profiles (odd/even). It would also be interesting to explore these rarer equidistant cases and the special conditions that lead to their consensus. Finally, the exploration of fragmentation and proving below a certain threshold that consensus is impossible would be another avenue of interest.

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Appendix A

Bounded Confidence Python Function

This function was designed to take an array of opinions (either random or symmetrical) and determine the set of opinions using (2) that were ε-close to a particular agent. It then took all values that were within epsilon distance of that agent, counted the number of agents considered, averaged the opinion set using these values and stored the calculation as an entry in the new opinion array. This is done for every agent in the original array to create 1-time step for the opinion profile. This function was used for a variety of different computations including the Monte Carlo Simulations, the opinion profile graph plots, time to consensus graphs and frequency of consensus for evenly spaced profiles.

```
import numpy as np
from numpy import random

# takes an opinion profile vector  and a designated confidence interval ep
silon and takes the
#average of neighboring opinions (other numbers in the vector) that are wi
thin a distance of epsilon
# producing a new vector called new_opinion_profile

def bounded_confidence(initial_vector, epsilon):
    new_opinion_profile = np.empty(initial_vector.size)
    for element in range(0, initial_vector.size):  #
        new_array = abs(initial_vector - initial_vector[element])  # deter
mine the "closeness" |xi - xj|
        counter = 0
        sum_elements = 0
        for each_element in range(0, initial_vector.size):
            if new_array[each_element] <= epsilon:
                sum_elements = sum_elements + initial_vector[each_element]
                counter += 1
        new_opinion_profile[element] = sum_elements / counter
    return new_opinion_profile

# Changeable Parameters; opinion profile, iterations, epsilon and
tolerance

number_of_agents = #Enter an "N" value
opinion_profile = random.rand(number_of_agents)
epsilon = #Enter a confidence value
tolerance = 0.5*10**-8
```

Appendix B

Monte Carlo Consensus Frequency Data

Monte Carlo Consensus Frequency Data																		
# of Agents	0.00001	0.01	0.05	0.1	0.15	0.2	0.25	0.28	0.3	0.32	0.34	0.36	0.38	0.4	0.42	0.45	0.5	Epsilon
5	0.00%	0.00%	0.00%	0.60%	0.41%	11.20%	22.20%	31.60%	38.90%	43.40%	51.00%	57.70%	62.90%	67.80%	73.50%	80.80%	88.50%	
10	0.00%	0.00%	0.00%	0.00%	0.30%	7.80%	25.40%	39.20%	52.60%	61.90%	72.30%	78.50%	84.60%	91.40%	93.30%	95.60%	98.60%	
20	0.00%	0.00%	0.00%	0.00%	0.20%	6.20%	31.40%	54.80%	67.40%	78.50%	87.90%	93.60%	95.50%	98.30%	99.00%	99.70%	100.00%	
30	0.00%	0.00%	0.00%	0.00%	0.10%	6.70%	41.30%	64.30%	78.00%	88.70%	94.30%	98.00%	98.60%	99.90%	100.00%	100.00%	100.00%	
40	0.00%	0.00%	0.00%	0.00%	0.10%	5.60%	42.30%	68.90%	86.40%	93.00%	97.80%	99.20%	99.60%	99.90%	100.00%	100.00%	100.00%	
50	0.00%	0.00%	0.00%	0.00%	0.10%	5.30%	44.70%	73.20%	89.50%	97.30%	99.00%	99.60%	99.90%	100.00%	100.00%	100.00%	100.00%	
60	0.00%	0.00%	0.00%	0.00%	0.00%	6.40%	47.90%	79.40%	92.70%	97.80%	99.20%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
70	0.00%	0.00%	0.00%	0.00%	0.20%	5.30%	52.50%	82.70%	95.60%	98.70%	99.70%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
80	0.00%	0.00%	0.00%	0.00%	0.00%	6.10%	56.80%	86.90%	96.70%	99.50%	99.80%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
90	0.00%	0.00%	0.00%	0.00%	0.10%	6.70%	59.20%	89.60%	98.60%	99.30%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
100	0.00%	0.00%	0.00%	0.00%	0.10%	7.00%	64.10%	92.50%	98.00%	99.70%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
110	0.00%	0.00%	0.00%	0.00%	0.00%	6.50%	66.00%	94.80%	98.90%	99.80%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
120	0.00%	0.00%	0.00%	0.00%	0.00%	6.40%	67.00%	95.40%	99.60%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
130	0.00%	0.00%	0.00%	0.00%	0.00%	7.00%	70.10%	96.50%	99.40%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
140	0.00%	0.00%	0.00%	0.00%	0.00%	8.80%	76.20%	96.80%	99.80%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
150	0.00%	0.00%	0.00%	0.00%	0.00%	7.30%	73.90%	98.40%	99.90%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
160	0.00%	0.00%	0.00%	0.00%	0.00%	6.10%	76.80%	98.40%	99.70%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
175	0.00%	0.00%	0.00%	0.00%	0.00%	8.60%	78.60%	98.10%	99.90%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	
200	0.00%	0.00%	0.00%	0.00%	0.00%	5.70%	81.20%	98.80%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	100.00%	

Appendix C

Time to Consensus Data

Time to Consensus (Ɛ = 0.2) Odd

Number Of Agents	Time to Consensus
101	40
201	71
301	103
401	134
501	167
601	199
701	230
801	262
901	294
1001	326
1101	358
1201	389
1301	421
1401	453
1501	485
1601	516
1701	548
1801	198
1901	612
2001	219
2101	676
2201	241
2301	739
2401	771
2501	802

Time to Consensus (Ɛ = 0.25) Odd

Number Of Agents	Time to Consensus Odd
101	10
201	8
301	9
401	8
501	9
601	9
701	9
801	9
901	8
1001	8
1101	8
1201	8
1301	8
1401	8
1501	8
1601	8
1701	8
1801	8
1901	8
2001	8
2101	8
2201	8
2301	8
2401	8
2501	8

Time to Consensus (Ɛ = 0.25) Even

Number Of Agents	Time to Consensus Even
100	9
200	10
300	9
400	9
500	9
600	9
700	9
800	9
900	9
1000	9
1100	9
1200	9
1300	9
1400	9
1500	9
1600	9
1700	9
1800	9
1900	8
2000	9
2100	9
2200	8
2300	9
2400	8
2500	8