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Towards a classification of continuity and on the emergence of generality

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Towards a Classification of Continuity and On the Emergence of Generality

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Abstract

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This dissertation has for its primary task the investigation, articulation, and comparison of a variety of concepts of continuity, as developed throughout the history of philosophy and a part of mathematics. It also motivates, and aims to better understand, some of the conceptual and historical connections between characterizations of the *continuous*, on the one hand, and ideas and commitments about what makes for *generality* (and *universality*), on the other. Many thinkers of the past have acknowledged the need for advanced science and philosophy to pass through the "labyrinth of the continuum" and to develop a sufficiently rich model or description of the continuous; but it has been far less widely appreciated how the resulting description largely informs our ideas and commitments regarding how (and whether) things become general (or how we think of universality).

The introduction provides some motivation for the project and gives some overview of the chapters. The first two chapters are devoted to Aristotle, as Aristotle's *Physics* is arguably the foundational book on continuity. The first two chapters show that Aristotle's efforts to understand and formulate a rich and demanding concept of the continuous reached across many of his investigations; in particular, these two chapters aim to better situate certain structural similarities and conceptual overlaps between Aristotle's *Posterior Analytics* and his *Physics*, further revealing connections between the structure of demonstration or

proof (the subject of logic and the sciences) and the structure of bodies in motion (the subject of physics and study of nature). The network of connections that exists between the two is shown to hinge on a particular notion of continuity, especially as this notion relates to the concept of generality (through the largely ignored notion of what Aristotle calls "suggenicity," or belonging to the same genus). This chapter also contributes to the larger narrative about continuity, where Aristotle emerges as an influential early proponent of an account that aligns continuity with *closeness* or relations of nearness.

Chapter 3 is devoted to Duns Scotus and Nicolas Oresme, and more generally, to the Medieval debate surrounding the "latitude of forms" or the "intension and remission of forms," in which concerted efforts were made to re-focus attention onto the type of continuous motions mostly ignored by the tradition that followed in the wake of Aristotelian physics. In this context, the traditional appropriation of Aristotle's thoughts on unity, contrariety, genera, forms, quantity and quality, and continuity is challenged in a number of important ways, reclaiming some of the largely overlooked insights of Aristotle into the intimate connections between continua and genera. By realizing certain of Scotus's ideas concerning the intension and remission of qualities, Oresme initiates a radical transformation in the concept of continuity, and this chapter argues that Oresme's efforts are best understood as an early attempt at freeing the concept of continuity from its ancient connection to closeness.

Chapters 4 and 5 are devoted to unpacking and re-interpreting Spinoza's powerful theory of what makes for the 'oneness' of a body in general and of how 'ones' can *compose* to form ever more composite 'ones' (all the way up to Nature as a whole). Much of Spinoza reads like an elaboration on Oresme's new model of continuity; however, the legacy of the Cartesian emphasis on local motion makes it difficult for Spinoza to give up on closeness altogether. Chapter 4 is dedicated to a closer look at some subtleties and arguments surrounding Descartes' definition of local motion and understanding of 'one body', and Chapter 5 builds on this to develop Spinoza's ideas about how the concept of 'one body' *scales*, in which context a number of far-reaching connections between continuity and generality (through his *common notions*) are also unpacked.

Chapter 6 leaves the realm of philosophy and is dedicated to the contributions to the continuity-generality connection from one field of contemporary mathematics: sheaf theory (and, more generally, category theory). The aim of this chapter is to present something like a "tour" of the main philosophical contributions made by the *idea* of a sheaf to the specification of the concept of continuity (with particular regard for its connections to universality). The concluding chapter steps back and discusses a number of distinct characterizations of continuity in more abstract and synthetic terms, while touching on some of the corresponding representations of generality to which each such model gives rise. The dissertation ends with brief discussion of some of the dominant arguments that have been deployed in the past to claim that continuity (or discreteness) is "better."

Contents

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Introduction

Motivation

Is the universe continuous or discrete? Consider just a small sample of possible responses: the universe is discrete since quantum theory and general relativity *together* suggest the existence of some minimum length; the universe is continuous since general relativity *on its own* suggests that spacetime should be a smooth manifold; the universe is a (discrete) computational system, say a universal Turing machine, operating in a discrete infinite time;[1] the universe is both, for just like existing mathematics, it is a complex blend of discrete and continuous phenomena; the universe is continuous but (most of) our models of it are (or should be) digital, discrete; the universe is neither (or, if forced to choose, perhaps discrete), for the "unscientific" or "theological" idea on which all of this rests—that mathematics is an infinite hierarchy of infinite sets—is simply a disavowal of the fact that all our notions of infinity (and by extension continuity) are fundamentally rooted in the integers, in counting, in the simple experience of finite iteration or "going on."

Everyone already has *some* idea of "continuity," or at least examples of phenomena that they are willing to entertain as being paradigmatically continuous. Perhaps you imagine things like fluid flow; the gradually changing tints of color of the sky; the variable intensities controlled by subtle alterations in pressure of the string-instrumentalist's fingers on strings, or

[1] Simply put, a universal Turing machine is basically just an abstract general-purpose "computer" of sorts, programmable to run any algorithm (as opposed to just carrying out special-purposes operations). More generally, Turing machines are just state machines—meaning that at any given moment the machine is in any of a finite number of states—that transition between states on the basis of instructions pertaining to the conditions of transition. A *universal* Turing machine can simulate the behavior of any (special-purpose) Turing machine.

the variations in sound produced by the theramin; or a certain "intuitive" notion of personal identity where the passages from any succession of related thoughts or perceptions is so smooth that we fail to perceive or notice the transition from one to the other. Continuity, then, is typically regarded as some sort of unbrokenness, some uninterrupted or smooth connection, even a merging of part into part. On the other hand, we equally seem to have some intuitive ideas of *discreteness* (and even, for each candidate continuous item, its discrete counterpart). Perhaps you imagine things like the separate vibrating particles of a fluid; the distinct gases that make up the earth's atmosphere; distinct colors such as black and white; the separate notes on a piano played by the digits of a keyboardist; the fact that certain events or perceptions in the "theater of our mind" can seem so sudden and unprecipitated, or so different from what came before, that they disrupt our otherwise seemingly stable sense of personal identity. Discreteness, then, is typically held to involve divisions and interruptions resulting in distinct, separate, or detached parts.

But to achieve an adequate conception of continuity—and a correspondingly adequate concept of its correlate, the discrete—is obviously far more difficult and involved than simply listing examples of things many of us are willing to take as intuitively fitting the bill, and then "pattern-matching." To do so would be to attempt to answer questions that go to the core of what the universe is with a poll. However, the difficulty in approaching this issue is not just that the variety of (frequently competing, but sometimes partially overlapping) conceptions of continuity, developed throughout the history of philosophy and mathematics, is truly astounding; or that, even adopting a single conception of continuity—and selecting one of the more precise conceptions of continuity that come from mathematics—it can happen that one and the same thing can be described as both continuous and discrete, or it can be regarded, at one level, as the paragon of continuity, and at another, as involving discreteness. While this variety is partly responsible for why the original question receives so many different, and often incompatible, answers, the underlying difficulty stems from how the issue of continuity-discreteness is, in many ways, so fundamental that at first it can seem impossible to understand what reasons and considerations could even be put forward in favor of certain conceptions over others. One is reminded of a moment in the ancient fragments left to us of Parmenides' part-cosmological, part-philosophical didactic "poem"—where, in the midst of stressing that *being* and *being continuous* are basically to be equated, we are asked "For, what kind of origin for [what *is*] would you look for?"[2] The drift of such a question is obviously different and more cosmologically-oriented in that context, but in even entertaining the question of how to best think of continuity, we get a similar feeling of having hit bottom and being lost as to how we might even begin to orient ourselves.

It is not that the above question is not well-posed, but it does seem to rest on a much more basic, and perhaps deeper, set of questions: how, in general, are parts determined *as parts*; how are those parts brought into mutual relationships such that they *hold together*; and under what circumstances do things that hold together *cease to hold together*? For instance: how do the component parts of a melody form a *single* melody; how can a line be *composed* of points or other parts; how is a species now *this* species, and then, at a certain point, *another* species; how are you the same 'you' even after undergoing radically different experiences? There is no single answer to such questions, but the highly saturated and protean concept of continuity forms a labyrinth through which all answers to such questions

[2]Coxon, *Fragments of Parmenides*, 8.5-6.

are forced to pass. In the history of philosophy and mathematics, developing answers to this nexus of questions has involved developing a rich and powerful concept of continuity.

There is another fundamental issue involved in the above question, namely the very notion of *a universe* or of a "whole" in which all else is included. In a sense, then, it is not that the question cannot have an answer, but that the one who responds will not really be able to extricate their answer to the question from their answer to the further question "and how is there *a universe*?" One typically finds that, in answering this latter question, the focus is on how "manys" in general compose to form greater unities or wholes, i.e., one specifies how to construct a universe "from the ground up" and only then tries to scale this to the "whole of wholes" (a "universe"). The battles fought here involving disputes over prior ideas about how, in general, things compose, over the particular ways of representing and relating the different manners in which parts are determined *as parts* and brought into mutual relationship, over how different unities can compose "greater unities" or come to form parts of still other wholes—these are where one observes different (and often competing) commitments and considerations regarding a particular conception of *continuity* (and when, correlatively, continuity breaks down or cannot be found, leaving discreteness).

Regardless of the model of continuity (or overlapping models) adopted, one must acknowledge (and be willing to attempt to explain) the many examples of systems comprised of discrete components that give rise to continuous behavior as well as continuous systems giving rise to discrete behavior or changes. There are many examples of continuity emerging out of the discrete. Fluid flow appears continuous (and is typically held to be described by continuous equations), yet real fluids consist, at an underlying level, of discrete particles in random motion. A movie or motion picture is a finite number of distinct still images, presented one after the other, yet when a certain threshold speed is crossed, the appearance of a continuous moving image is produced for the viewer. Our eyes scan the horizon *cut by cut*, taking in discrete sensory inputs, and our perception system synthesizes these discrete inputs, interpolating from such inputs and "perceptually completing" any missing information. Finally, when viewing a photograph or television, one can have the impression that everything one sees in the photograph or on the screen is continuous, yet really there is a point lattice where individual cells are being selectively illuminated by something like a "spotlight" scanning across the grid. Examples of this sort might suggest the idea that, in fact, all of nature is something like a television screen: whenever one observes continuity, there is really only discrete particles. On the other hand, there are many examples of discreteness emerging from continuity. A very conspicuous example of this occurs in phase changes, where, for instance, water changes temperature continuously until, suddenly, this gradual succession is interrupted by a change in state (such as when boiling water turns from a liquid into a gas, or when cooling water suddenly freezes and solidifies). On the basis of such changes in state, one might be inclined to conclude that, at least some of the time, nature does "make leaps." We stretch a rubber band, deforming it continuously, and then we stretch it a little more, still continuously, and suddenly it snaps and breaks. Moreover, one can have a continuous system, such as that modeled by a continuous curve or "hump" over which a ball is rolled continuously, in which discrete outcomes or behaviors are observed as one varies the initial conditions in a continuous fashion.

At the level of common models, this same back and forth can be found. On the other hand, there can be a number of difficulties in passing from the continuous to the discrete (and vice versa), for instance as one observes with the initial difficulty of getting AI systems to translate visual input (continuous) into lists of objects (discrete), or with the incapacity of classical (discrete) logic to deal with certain phenomena involving continuous variation. And there are even various curious "mixings" of the continuous and discrete to be found in the historical register, for instance as seen in the somewhat ironic fact that the word "calculus," which is now basically the "poster child" of continuous systems, is derived from the word *calx*, a small pebble or stone used for counting and doing discrete calculations.

Whatever notion of continuity-discreteness one adopts, it would seem to be naive to expect the entire universe, in every way, to be continuous *or* discrete. Across many (if not all) of the different senses of continuity-discreteness, it appears to be a fact that one can find continuous phenomena that evolve from discrete components, just as there are discrete phenomena or behaviors that evolve from systems held to be fundamentally continuous. At the very least, then, if one's account is any good, we expect that the answer to what makes for continuity and how this allows for some sort of viable procedure of "composition," would have to be rather nuanced and flexible. In short, what makes for the "unity" of pluralities in general cannot be something entirely invariant or static; one must deal with the fact of the many changes such unities can support—from those to which they are robust, to those that eventually destroy or transform their principle of unity. Many unities appear to change "internally"; others can "combine" to form greater unities; others can have their characteristic principle of unity "emulated" by different systems; still others are subordinated to new principles of unity.

In natural and human systems alike, there is no shortage of relative invariants of all sorts, subversions and renegotiations of prior domain restrictions, the formation of robustness or thresholds of resistance to certain changes, the emergence of dynamic mechanisms for filtering and determining what is "inner" (as which parts can be appropriated or integrated) and what is "outer." Wolves become two hundred breeds of dogs; a crystal takes disordered material and spontaneously organizes it by incorporating it into its own structure; a cell membrane forms with selective permeability, filtering and controlling what can enter and exit, regulating the transport of foreign materials; a trait is canalised;[3] *E. coli* bacteria assess the perimeter of a wide range of different carbohydrate molecules and, finding the display of a particular active site, subsequently swim upstream along a sugar gradient; two numbers are added very far along in a non-terminating decimal number, resulting in carries in the entire sequence that propagate arbitrarily far; the total values of various quantities, like electric charge, are conserved throughout the evolution of a closed system; popular motives from human musical history can be found in bird songs; a move of a single stone in a game of Go propagates complex, cascading global (non-local) changes across the board.

In all sorts of systems capable of organizing, filtering, redistributing, and transferring materials and resources, one finds the ongoing and flexible construction and negotiation of boundaries (as in the cell membrane example); one finds systems in which certain local changes can propagate arbitrarily far through the system, resulting in global effects (as in the number and Go examples); one finds the migration, emulation, and capture of blocks of behaviors across (occasionally very) distinct systems (as in the bird song example); one finds robustness against certain perturbations, plasticity with respect to others (as in canalisation); one finds foreign material being reorganized according to certain structural demands of the

[3] Originally a term from the biological and genetic sciences, canalisation refers to a process found in many systems that describes thresholds of sensitivity to certain perturbations, beneath which differences in certain parameters do not lead to bifurcations. It is sort of a measure of how much change in the environment (or genotype) something can stand before its behavior (or phenotype) must change.

system ingesting that material (as in the crystal example); one finds habits of behavior connecting a *generic type* of shape with a *typical* action (as in the *E. coli* example); one finds relative constants in closed systems that do not fundamentally depend on the *individuals* that pass through the system (as in the closed systems example); one finds the internal differentiation and branching of a genus (as in the wolf-dog example).

If one attends closely to these sorts of (otherwise rather distinct) examples, one finds that there are all sorts of *relative invariants*, as well as displacements and renegotiation of such invariants. One finds that there are local changes giving rise, gradually, to more and more global changes, and that certain generic types or blocks of behavior come to be associated with certain typical actions and effects. And it is not clear how any of these sorts of things could be explained in terms of changes to *individuals* or already *individuated beings*. It seems like approaching such phenomena in terms of atomized individuals (or already individuated things) comes "too late" to be useful or to fully capture what is really going on here.

If not *individual*, then, what is the right "level" (or conceptual tools) with which to approach such phenomena? Much of the philosophical tradition would have us believe that, if not individual, then such things may perhaps be described as participating in *universality*. And there does indeed seem to be some sort of movement towards the more *general*, or transformations occurring at levels that are "above the heads" of individuals, in most of the above instances. But most of the usual conceptions of universality seem just as unable to explain the above sorts of phenomena; moreover, those conceptions are usually content to *assume* the individuals they seek to explain. They typically seek to understand what makes something a 'one' through the individual (or some aggregation thereof), instead of knowing the individual through that which "makes one." Under the shadow of what I could call the "standard universal" of much of the history of philosophy, philosophers have not, in general, been inclined or equipped to take seriously such phenomena and changes as those mentioned in the previous paragraph, let alone to construct entire ontologies capable of incorporating them. Moreover, if at least some "universals" did not already form some sort of continuum, it is not difficult to see that one could not hope to account for the evolution or changes of individuals, for how certain realities are the locus of internal differentiation and can participate as (at least partial) causes in their own transformation. For much of the tradition, under the shadow of the standard universal, the phenomenon of generality has often been reduced to a "mental operation" and then explained in terms of one (or some combination) of three main descriptions: (i) as a lack of specificity and an abbreviation of enumerated individuals; (ii) as a predication of a common property taken to be a means of recognizing or naming an object as the same again; and (iii) as a saving of cognitive (or other energetic) resources, i.e., a sort of economization.[4] Each of these representations of generality—while useful in very restricted contexts—is deficient when it comes to capturing the full wealth of the phenomenon of generality. It would be useful to offer a few details regarding these three characterizations that I am calling "standard universality," before going on to discuss generality more broadly (as well as its connections with the concept of continuity).

$Universal_1$ (lack of specificity; abbreviation): here, universality is regarded as the result of an *aggregation*, specifically as an aggregation of ready-made, numerically distinct

[4]There are certainly other relevant characterizations, perhaps equally worth calling "standard"; but it would be unnecessarily distracting to discuss these at greater length in the present context.

individuals, such that (a) those individuals are *externally related* to one another and to that which joins them under the principle of unity represented by the universal; (b) no determinateness in the individuals is *explained* by the universal, i.e., each individual is just "yet another instance" and thus just like any other; (c) no allowance is made for any internal variability or differences in degree between whatever is joined under the universal; (d) all determinations not directly pertaining to the identification of distinct individuals as *exempla* are purposefully removed or "forgotten"; and (e) marks pertaining to the provenance of the universal are eliminated. As such, generals can be no more than "mere abstractions" for they contain no information about the interrelations of the individuals, their sole and uniform function being to indifferently *aggregate numerically distinct individuals.* On this account, all universals are regarded as "alike," and any claims to certain differences in how one universal captures an interval or range of beings, as opposed to the interval spanned by another, are ignored. Such uniformity allows for the productive linking of distinct predicative generals (as in certain classical developments of logic), but pays the price of not being able to coherently account for different "levels" of universality. (One decides the "domain of discourse" and that is the end of it.) In *assuming* its individuals—and in failing to establish anything more than an external relation with its universal—this approach must ultimately give way to the notion that any given universal could, at least in principle, get replaced by, or reduced to, the *complete enumeration of its individual examples.* But given that such enumeration is impractical (or even impossible) in many cases, generalities are ultimately held to be some sort of *convenience*, something like a shorthand or abbreviation. Regarded thus—as an abbreviation justifiable only on the grounds of "convenience"—$universal_1$ is complicit in its undoing at the hands of nominalism.

Universal$_2$ (common properties): here, universality is *predicative generality*, i.e., predication of a *property* common to a number of individuals that are held to *resemble* one another in this respect. This universal-type is closely connected with *universal*$_1$. This predicative universal characteristically tells us nothing about the interrelationships between the individuals that support this property, but only isolates a static attribute and predicates it of many numerically distinct individuals. Moreover, it cannot account for the *genesis* of those universals or for the genesis of that "common" attribute as applying to those individuals in particular. And to the extent that it *does* address this issue, it can only say that the "similarities"—that, in turn, are supposed to ground the act of common predication—emerge after having been "abstracted from" multiple particulars.

One problem is that the 'particulars' of this universal-type would appear to be, from the beginning, already *more than* particular to the extent that each particular is comprehended or perceived *as* something, in which comprehension the very generality allegedly reached by subsequent 'abstraction from individuals' already appears to be at work. Another complication: to the extent that this predicative generality understands universals to be the result of some sort of process of *induction*, it can only misunderstand this process—for it would describe induction as the extension of a property to an *entire* collection of individuals simply from the observation that it belongs to *some*. But it hardly ever has any account of how such an "extension" is supposed to work. As predicative generals constitutively erase the successive and tentative operations by which individuals are combined and unified under a universal—which acts in themselves, at first, often have nothing to do with *asserting/denying* a common property—there can be little recognition or awareness of the prices paid, and currencies used, in the growth of its generals.

$Universal_3$ (economization): here, while the focus is on the *genesis* or the evolution of generalities or universality, this is explained in terms of a "saving of (usually cognitive) resources." Generals are taken to be the result of induction from specimen cases, resulting in the generation of a rule that can *interpolate, extrapolate, and predict* new cases from previous cases.[5] But one way of seeing the limitations of this characterization is to consider how, at least in the context of certain common learning problems, it has been shown that in the absence of other evidence, if two classifiers x and y cover the same cases from an initial training data set and x is a generalization of y, then there is a higher probability of x *misclassifying* previously missing or withheld cases (once these are reintroduced).[6] There is further evidence that, in certain contexts at least, *predictive accuracy decreases as generality increases*, and that it does so regardless of the means by which generality was created, i.e., regardless of whether the generality is introduced by increasing or decreasing the complexity of the classifiers. This is of course not in itself a counter-argument to the main idea of $universal_3$; but it is one way of beginning to cast some doubt on the plausibility of any simplistic explanation of the genesis of generals terms of "utility," as "saving resources" by virtue of supplying some predictive gain. There might indeed be some "trade-off" at

[5]I see nothing wrong with this last characterization of generality in its own right. I only take issue with the thesis that such generals emerge in order to "save resources." But this universal-type does offer something not usually noticed (even if these matters are often misinterpreted): $universal_3$ correctly apprehends both (i) that generals cannot be understood independently from the *special operations by which they are produced*; and (ii) that from their humble beginnings all the way through their many revisions, confirmations, domain adjustments, and reorientations—generals are best characterized as indexed to certain contexts, contexts that are themselves not fixed once and for all but can vary.

[6]See, for instance, Geoffrey Webb, "Generality is more significant than complexity"; for a related study, see Nick Chater, "The Generalized Universal Law of Generalization." In Webb, more general classifiers are generated by two alternative methods: conjunct deletion (decreasing rule complexity) and disjunct addition (increasing rule complexity). Incidentally, Webb notes that this further suggests that, against Occam's Razor, "generality, not complexity, should be the determining factor in selecting between classifiers with equal empirical support."

play here, where certain "metabolic prices" are avoided by virtue of this greater generality, gains that can offset the possible losses in predictive accuracy. However, at least without further clarification, it is simply not clear how generality as an "economization of (cognitive) resources" is ever supposed to have proven *adaptive*, if predictive accuracy can *decrease* as generality increases.[7] Another issue concerns the fact that while such an explanation may very well apply to certain ways human beings have developed the use of certain concepts, examination of a wider class of natural systems makes one more suspicious of the idea that in enlarging the scope of influence of certain behaviors or traits or ideas, lifting previous constraints or domain restrictions, or incorporating previously missing elements—in short, in becoming more general—*all* natural or human systems can always be described as engaging in an "economization" of resources. In short, even ignoring certain inherent complications, it is not clear whether this "economization" account could be ported to other systems (which is something we would, at least in principle, like to be able to do).

Individuals, species, forms—these things are not ready-made or given once and for all. But even if there were such givens, individuals *as individual* do not seem able to account for any of the sorts of important and pervasive evolutions and changes described earlier. The various movements towards generality as involving changes whereby constraints and scope or domain restrictions are variously lifted and transplanted in such a way that the field of influence of those entities subject to such constraints becomes broadened—these changes do not seem to be explicable at the level of individuals, or even to involve the simple possession

[7] Of course, not every development is an adaptation; and, in general, adaptations are not optimally designed. Another possibility is that generality involves some other savings or re-allocation of resources, say by requiring the storage or consideration of fewer constraints, allowing energy or resources to be dedicated to other matters.

(or absence) of a static property predicated uniformly of all of its constituent individuals. For its part, the standard universal sovereignly presiding over its "individuals" is simply too blunt an instrument to even register, let alone capture or sift through, many of the relative and shifting invariants of the sort discussed above. And to the extent that the standard universal does capture certain aspects of the phenomenon of generality, it still seems to be something that must first be grounded in a more complete account of what Duns Scotus would have called the *fundamentum universalitatis*, the ground or basis for universality in nature, the real commonness that does not reduce to a logical question of predicability but forms the ground for the very possibility of predication. Moreover, to take just one aspect of generality as a movement whereby, e.g., local changes come to achieve global effects, this does *not* seem to be usefully explained or accounted for by the standard universal. Yet in forging passages from the local to global, such transformations and processes do seem indisputably to involve something like a lifting of previous scope or spatial restrictions, allowing for a cascading communication of motions and determinations between previously remote parts such that the principles of unification of the given complex or network of parts are importantly modified to now include more parts. This is something that often can neither be reduced to an exhaustive account of the parts individually, nor explained by the limited tools of the standard universal.

One could certainly just dismiss this sort of observation, or one could continue to content oneself with poking at the carcasses of the standard universal. But such responses seem indefensible. There seems much more reason to acknowledge that there are developments and transformations that take place *above the level of individuals*, happenings that can neither be explained by the standard universal nor reduced to the many mysticisms of "individuals."

It would be more reasonable, at this point, to wager that generality is simply a far more dynamic, complex, and internally differentiated phenomenon than the standard universal of tradition would suggest, a phenomenon worth "updating" our philosophical toolbox to accommodate.

From the broadest of standpoints, the initial wager of this dissertation is basically that starting to better understand the phenomenon of generality will ultimately have to involve attaining a better understanding of (our characterizations and commitments regarding) the conditions making things *continuous*. I claim that the particular manner of establishing when and how there is *continuity*—from one moment to the next, for certain parts in relation to others, for a given region of beings, etc.—is what forms the basis of the *emergence* of generals. Generals are dynamic and manifold economies trading in the universal currency of continuity conditions; or, to change the metaphor, generals speak in the language of continuity. There is no unity in multiplicity without some aspect of generality; but there are no generals without continuity conditions. Moreover, once understood as determined by their specific continuity conditions, generals or universals will more readily be seen as involving both relative invariance and internal variability, plasticity, robustness with respect to certain perturbations, and sensitivity with respect to others. The dependence of an understanding of the phenomenon of generality on specific continuity characterizations is enormous and pervasive, if somewhat subtle and frequently overlooked by most traditions.

Aside from the many intrinsic interests of developing such ideas, there are good reasons for resisting the prolongation of the prevalent habit of thought that dogmatically insists that the becoming of generals can be reduced to a problem of psychology, epistemology, or even that it is a phenomenon or process that in principle could only pertain to human systems. This approach seems not only not to advance thought, but it would appear to fly in the face of the phenomena, in addition to leaving us with an impoverished logical toolbox and a veritable avalanche of inconsistencies in the application of key scientific concepts. Since I am claiming that the gradual determination of generals is a process overwhelmingly pre-determined by (or inherited from) prior commitments and characterizations regarding *continuity*, a robust model of generality (strong enough to endure its extension beyond the confines of epistemology) will have to emerge organically out of a careful and more complete picture of the many different faces of continuity. This latter task is accordingly the main focus of this dissertation. The former task—of developing a more sophisticated and accommodating notion of generality or universality on the basis of a better understanding of continuity—will have to be postponed or left to the combined work of others. One of the consequences of this embedding of the problem of the formation of generals in the larger space of the protean concept of continuity will be to help set the stage for subsequent constructions of a more dynamic, nuanced, and exacting theory of generality.

Incapable of believing any longer in its impoverished abstract universals, but equally unable to turn its back on the phenomena so much as to assert the total victory of unqualified nominalism or endorse the dogmatism and implicit mysticism of any form of "atomism," much of philosophy has been forced into a sort of unconscious trafficking in intermediate generals. This project aims to contribute in a small way to raising this to self-consciousness by refining and relativizing the concept of universality. What this amounts to in the end, I believe, is the partial and initial development of something like an *objective logic.* As I understand it, an *objective logic* would begin from the realization that the modes of connection of parts or individualized units of systems do not (always or for the most part) arise from (cannot be explained by) the nature of those parts alone. The nominalists have always been right that these connections cannot just be *assumed as given* or taken for granted. However, in concluding from the failures of those systems that had been content to *assume* such connections that *there simply are no generals in nature*, nominalism abandons the phenomena for an equally troubling form of dogmatism and mysticism. An objective logic would forge a viable path between the impoverishment of standard universalism and the inconsistency of a nominalism of unregulated, swarming "individuals." It would begin by laying out the various ways connections (between ideas, things, systems, whatever)—on which subsequent representations of generals are based—are established in terms of characterizations and commitments regarding continuity.

Of course, there have been thinkers in the past who have been suspicious of both the monopolization of generality by the standard universal and the blind dismissal of the pervasiveness of generality in phenomena, and who have accordingly attempted to construct partial refinements in the concept of generality (and who, naturally, developed strong ideas on the nature of continuity). One could name Aristotle's (ignored) treatment of the phenomenon of "suggenicity," Scotus's fusion of intensive magnitude and essence and his "common natures," Oresme's intensive or "figurational" geometry, Spinoza's "common notions," Hegel's concrete general, Peirce's category of "thirdness." However, there has been no concerted effort to bring philosophy's ancient power of categorical and classificatory thinking to bear on the concept of continuity, so as to start to allow for a better appreciation of how disputes over competing theories of generality largely boil down to differences in continuity conditions. These isolated attempts to put forward a more nuanced account of generality thus mostly remain adrift in a massive sea of mostly inconsistent pronouncements, far from the tiny and disintegrating glacial islands of universality and individuality.

One of the most important ways of approaching a study of the pronounced influence of continuity-types on the elaboration of a notion of generality is to pay attention to the relation between quantity and quality—call it the logic of quantity and quality—specifically at the moment of their differentiation where the category of *variable quantity* is first delimited. The concept of variable quantity is not only where borders and negotiations between extensive and intensive quantities are subsequently unfolded, but where one can also appreciate most clearly that *continuous variation* amounts to a form of generalization. Objective logic, as I understand it, does not begin with essences, truth, being, substances, or constants (atoms or individuals); rather, it is constituted from beginning to end by the manners of constructing and defining *space in its various modes of determination and coherence through its distribution into quantity and quality.* How is quantity differentiated from quality? What is the nature of the relations between the two differentiated realms? Are there definite patterns of mutual relations between them? What is the nature of the dependence of each of intensive and extensive quantities on the *space* over which the quantities are distributed? Can the notion of continuously variable quantities viably be seen as a generalization of the notion of a constant quantity? Or are constants something of a limit of continuously variable quantities? If so, how do continuously variable quantities arise and how can constant quantities be made to emerge on the basis of variable quantities? How does "quantification" depend on the spaces or domains over which it ranges? These are just some of the basic questions orienting such a logic.

Boundaries are unfolded according to the model of continuity—how a plurality forms a unity or coheres into a 'one'—with which one is operating. On the basis of this model, spaces are determined (regions where certain things "hang together" and others do not), and it is in relation to these spaces that extensive and intensive determinations of "objects" unfold. Accordingly, all determinations concerning what it is for something to be *discontinuous* or *discrete* are anchored in (and ultimately propagated by) a corresponding model of the continuum or a continuous space. However it is determined, discreteness always carves up a space that is not just any space but is itself determined by a particular mode of guaranteeing continuity between parts. As such, the dialectics of continuity-discreteness is in no way to be reduced to an *opposition*—the two are always complementary, and once one knows how someone determines the continuity of things, one usually more or less immediately knows what is discrete for them. When we speak of a model of continuity, we can usually speak in the same breath of a corresponding manner of discretizing a region of space.

Familiarity with the details of the many different existing models of continuity-discreteness is enough to cure one for good of the default setting of the majority of us: a sort of naive realism with respect to the generalities we employ and engage with every minute of our lives. And to the extent that we do occasionally explicitly acknowledge more than one manner of establishing continuity—manners we ordinarily unconsciously accept as securing the passage from one isolated being to another, thereby preparing the way for generality—these are usually confused and conflated with still other models that have not even been raised to the level of self-consciousness. Accordingly, this dissertation begins by acknowledging the fact of *the immense proliferation of distinct models of continuity*, and takes as its primary aim the investigation, articulation, and comparison of a variety of concepts of continuity as developed especially throughout the history of philosophy but also a part of mathematics. This dissertation is above all else designed to force more careful thinking about some of the more far-reaching or deep-seated ideas about how things "hold together," just what this depends on and involves in different cases, and where such different commitments lead us.

While the final two chapters approach things from a more conceptual and abstract standpoint, the first five chapters are firmly anchored in certain moments in the history of philosophy, around which moments the structure of a larger narrative will emerge. While some readers may not need convincing of this, it is worth remarking that conceptual disputes and commitments are often strongly determined by certain aspects in the history of that concept's development and by the story ordinarily told about that, even (or especially) in the absence of explicit awareness of (or engagement with) that history. I believe that the story of the concept of continuity is one that still needs much more attention: many gaps need to be filled, other aspects of the story need refining, and certain received truths deserve to be challenged or complicated. Accordingly, a large portion of this dissertation is devoted to contributing to a more complete history—both emphasizing and interpreting certain overlooked moments and aspects of the problem, as well as complicating the interpretation of other (more conspicuous) moments.

Before summarizing the content of the chapters of this dissertation, I can compactly present my understanding of the overall thematic unity of this dissertation in the form of a parodic syllogism of sorts (full of seeming tautologies): To be is to *continue* to be.[8] To

[8]Spinoza would say: "The power or *conatus* by which each singular thing strives to persevere in its being is nothing other than the actual essence of the thing itself" (Spinoza, *Ethics*, III.P7).

continue to be is to propagate or multiply one's being.[9] To propagate one's being is to participate in the becoming of something general.[10] Therefore, the becoming of generals unfolds according to the particular manner by which beings *continue to be*.

Overview of Chapters

The first two chapters of this dissertation are devoted to Aristotle. In many ways, Aristotle's *Physics* is the *Grundbuch* of continuity—the foundational book—that, even in its omissions and oversights, came to rule over so much of what has since been thought about continuity. In the past, discussions of Aristotle's thoughts on continuity have focused almost exclusively on a small number of select passages from the *Physics*. The first two chapters of this dissertation begin from the idea that Aristotle's efforts to understand and formulate a rich and demanding concept of the continuous reached across not just most of the *Physics*, but across many of his investigations more widely. Together, these two chapters aim to better situate certain structural similarities and conceptual overlaps between Aristotle's *Posterior Analytics* and his *Physics*, in particular, further revealing connections between the structure of demonstration or proof (the subject of logic and the sciences) and the structure of bodies in motion (the subject of physics and study of nature). The network of connections that exists between the two is shown to hinge on the concept of continuity, especially as this notion relates to the concept of generality (through the largely ignored notion of what Aristotle calls "suggenicity," or belonging to the same genus).

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[9]Spinoza would say: To strive to persevere in our being is just to strive to *augment* our power to act, to be affected with a greater number and diversity of affections while preserving our characteristic 'oneness'.

[10]Spinoza would say: To augment our power to act is just to come to compose with, and come to understand oneself to be, a greater part of nature.

Chapter 1 is dedicated to the *Posterior Analytics* in particular and focuses on Aristotle's arguments against the possibility of infinite demonstration; his arguments for positing certain indivisible principles (together with his attempts to use these to guarantee the coherence of demonstration as a whole); and his arguments against "genus crossing" from one demonstration to another (which heavily involves his notion of suggenicity and further allows him to secure the 'oneness' of each distinct science and its underlying subject matter). Against the traditional interpretation that regards Aristotelian continuity as essentially involving (infinite) divisibility and against the overly simplistic interpretation that Aristotle is an "atomist" when it comes to demonstration or logic but "saves" change in his response to Zeno in the *Physics* by precisely dismantling the atomistic account, I identify and articulate a structure in the *Posterior Analytic*'s approach to demonstration that re-appears almost identically in the *Physics*: namely that the characteristically indivisible limits of continua act at once to make the whole of which they are the limits internally continuous, while externally discretizing them. Taking this reasoning to the limit, the *Physics* works up to the conclusion that only the cosmos as a whole moves continuously, in the strictest sense, and that it does so only on account of the action of the first and indivisible unmoved mover. In this way, something that is characteristically not in or a part of nature, and is characteristically indivisible and unmoving, becomes the sole guarantor of the continuity of nature as a whole and its motions; the first mover is the first thing that is not itself movable but forms the basis of all motion. In the *Posterior Analytics*, the coherence and necessity of each given demonstration (and the continuity between the propositions out of which they are built and their conclusions) is secured by the existence of immediate, indivisible principles. Taking this reasoning to the limit, the integrity of the entire enterprise of demonstration (and, so, scientific knowing as a whole) is secured by the existence of a kind of knowing, namely *nous*, that knows the indivisible principles and knows them "immediately," and stands as the first kind of knowing that is outside the scope of what is demonstrable but which forms the basis of all demonstration.

Chapter 2 is an extended exposition and re-interpretation of key passages of the *Physics* that treat of continuity, in the course of which the argument is made that, in the most important sense, continua are made continuous through indivisibles—and in this way continuity for Aristotle is ultimately aligned with finiteness and completion, and not primarily with infinite divisibility (as many interpreters have maintained). A number of connections with generality are also discussed. This chapter also contributes, in an especially pivotal way, to the understanding of the larger history of continuity (and its connection with generality), as Aristotle here gradually emerges as one of the most articulate early proponents of a massively influential account—namely, one that aligns continuity with *closeness*, making it fundamentally a matter of *relations of nearness*. This alliance between continuity and closeness will come to dominate much of the subsequent thinking of continuity and is an alliance that, as the story unfolds, will be importantly challenged by some of the remaining figures and theories discussed in this dissertation.

Chapter 3 is devoted to Duns Scotus and Oresme, and more generally, to the Medieval debate surrounding the "latitude of forms" or the "intension and remission of forms," in which concerted efforts were made to re-focus attention onto the type of continuous motions hitherto mostly ignored by the tradition that followed immediately in the wake of Aristotelian physics: qualitative motions. Here, in contrast with the traditional account of forms as characteristically invariable and indivisible, things like age, ripeness, loudness, color, and charity are understood to be qualitative or "formal" alterations that necessarily involve intermediary states and internal variability, and support changes in degree or "intensity." In this context, the traditional appropriation of Aristotle's thoughts on unity, contrariety, genera, forms, quantity and quality, and continuity is challenged in a number of important ways, reclaiming some of the largely overlooked insights of Aristotle into the intimate connections between continua and genera. In the first two chapters, it will be seen how, on a more considered reading of Aristotle, within each given genus, spanned by the interval formed from the "extreme contraries," there must be "in-betweens"—and so a genus emerges as something of a continuum (and, indeed, as something that supports a range of continuous variations). In the Medieval period, there were some, like Scotus, who took up the challenge of more systematically accommodating how certain realities that were regarded as "general" were continua. The Aristotelian account of both continuity in general and alteration in particular had set out a number of such problems for which the concept of intensity or *intensive magnitudes*—emerging out of the fact that certain qualities could become more or less of themselves—was to become, in the hands of certain Medieval philosophers, the principal solution. Scotus took special interest in the main question concerning the increase and decrease of qualities, in the course of which he advanced a number of subtle and profoundly transformative ideas. Perhaps the most transformative of these was his notion of a "transferred" sense of quantity, on the basis of which he would develop his formal and modal distinctions, and the concept of intensive modes as "inseparable from the nature of a thing." The "formalities" and qualities considered by Scotus emerge as intervals of variation, as admitting of degrees, and also as providing a kind of unity to these degrees. As the continuous variations in degree modified a certain "formality" and so were held to be "intrinsic" to it and unified by the *ratio* of that form, Scotus's ideas on these matters would further pave the way for a more nuanced understanding the phenomenon of generality and the traditional "problem of universals" (via his "common natures"). These ideas would be pushed to their extreme by Nicolas Oresme, a half-century later, in his radical attempt to geometrize the measure and comparison of intensities with his "figuration of qualities."

In addition to his profound realization of certain of Scotus's ideas concerning the intension and remission of qualities, Oresme initiates a still more radical transformation in the concept of *continuity*. This chapter argues that Oresme's efforts are best understood as an early attempt at freeing the concept of continuity from its ancient connection to closeness and extrinsic relations of proximity. Oresme first develops an incipient formalism for the representation of all sorts of intensities by continuous geometrical "figurations," i.e., in terms of (compositions of) lines and curves. On the basis of this, he attempts to develop a classification involving certain distinct generic types or species—corresponding to certain generic features of the figures—describing constraints on how the quality in question may vary in intensity. In this way, Oresme can begin to discuss the ways in which very different underlying "subjects" or qualities can change intensively in comparable ways. This leads ultimately to his theory of "concord" (or "consonance") and "dissonance" of distinct beings, thought through the conformity (or lack thereof) of their figurations and their respective "ratios of intensities," compatibilities that were thought to correspond to mutually compatible operations and similar ways of being "affected." Oresme grounds a way of thinking systematically about certain generic or "typical" features emerging in the characteristic way forms or qualities *change in intensity*, rather than in terms of some invariant or static property thought to belong (or not belong) to the subject or substance underlying or supporting that form or quality. This amounts to a picture of nature as shot through with morphological continuities and discontinuities, grounded ultimately in a notion of certain *generic types* of changing in intensity—which commonalities can be found even when the underlying qualities or subjects being changed are very different or far from one another—and in a further notion of how well (or poorly) these variation types "compose" with one another. In these two ways, continuity can now be treated as (1) an intrinsic property (of a single given quality or form changing in intensity and supporting a range of degrees); and, in passing to the relations between the distinct generic types such intensive changes describe, as (2) a matter of morphological "conformity." Following Oresme's own decisive observation of how this general account can be seen rather clearly in music, on the continuum describing the quality of sound, the *closest* (in the sense of the most proximal) sounds are in fact usually the most dissonant or "furthest" from one another (in terms of any structural conformity between the ratios of intensities describing the two sounds); the ratios of intensity with a "greater conformity" in the generic shape of their variation are those that can best be combined with one another into a composite "harmony" (and this remains the case even when the entities or bodies realizing these qualities are spatially far from one another). It is such relations of "conformity" between the generic types of intensive variation, instead of closeness in the sense of proximity or contiguity of material parts, that determines the true "measure" of the degree of continuity and "natural friendships" found between different sounds (and, more widely, between different entities throughout nature). This "freeing" of continuity from closeness, lifting it towards a more structural account in terms of relations of conformity between the shapes and ratios of the various generic types describing different patterns of variation in intensity, is perhaps the single greatest and furthest-reaching transformation of the concept of continuity (one that gets taken even further by Spinoza).

Chapters 4 and 5 are devoted to Spinoza's powerful theory of what makes for the 'oneness' of a body and his theory of how 'ones' *compose* to form ever more composite 'ones' (all the way up to Nature as a whole). With these ideas, rather than in his more explicit discussions of the (in)divisibility of extension and the nature of the (mathematical) continuum, Spinoza presents his most advanced and nuanced ideas on the nature of continuity (and also connections to generality). I believe that his true theory of continuity, wherein his greatest contributions to the understanding and advancement of this concept are to be found, lies buried deep within his "physics," specifically as developed in his rather involved account of what makes a body 'one', one that is robust to all sorts of changes while capable of being destroyed or subsumed by others, and how 'ones' compose (to form more and more composite 'ones'). In Spinoza's hands, continuity becomes even more "structural" and less tied to "closeness" than it was for Oresme. However, I understand Spinoza to play an absolutely pivotal role in attempting, in the most considered view of his project, to *reconcile* aspects of the tradition of continuity-as-closeness (that began with Aristotle and were still present in Descartes' treatment of local motion) with the Oresmian model of continuity-as-conformity-of-ratios. Much of Spinoza reads like an elaboration on Oresme's new model of continuity; however, the legacy of the Cartesian emphasis on local motion makes it difficult for Spinoza to give up on closeness altogether, and so in many ways he can be understood as striving to re-unite aspects of these two models. For this and other reasons (to be discussed in these chapters), a large portion of Chapter 4 is dedicated to a closer look at some subtleties in Descartes' definition of local motion and 'one body'.

Chapter 5 builds on this to take a closer look at Spinoza's theory of what makes bodies 'one' and how the notion of 'one body' can be *scaled*—all the way up to Nature as a whole. It is here, in the development of a notion of compositionality and how the concept of 'one body' scales, that one finds not just his most sophisticated thoughts on the nature of part-whole but also the seeds of a number of very far-reaching connections between continuity and generality (through his *common notions*). All of these ingredients together crystallize into what I describe as one of Spinoza's most profound ideas: that *continuity plus self-similar composition equals generality.*

Chapter 6 leaves the realm of philosophy and is dedicated to the contributions to the continuity-generality connection from one field of contemporary mathematics: sheaf theory (and, more generally, category theory). The main premise of this chapter is that, in a uniquely forceful way, sheaf theory enables us to start to clarify the still poorly-understood connections between the phenomenon of generality on the one hand and continuity on the other. The major aim of this chapter is to present, in abbreviated form, something like a "tour" of the main contributions of category theory and sheaf theory to the specification of the *concept of continuity* (with particular regard for its connections to the phenomenon of generality). More broadly, in the history of mathematics, I believe one can perceive a gradual "weaning" off of models and formulations of continuity in terms of "closeness" towards more and more structural and morphological accounts of continuity, after which there are some efforts to partially reconcile the two approaches—preserving what is best in our more "intuitive" or "tactile" understandings of continuity (as fundamentally a matter of the behavior of "close things" or local interactions), while benefiting from the power of the more "structural" and "morphological" formulations. I describe sheaf theory as embodying a particularly decisive example of such a "reconciliation." While the tools and concepts discussed in this chapter will initially be unfamiliar to most philosophers—though they will be well-motivated and progressively discussed at various levels of accessibility—the problem is one that will be very familiar. It has to do with how generality first emerges out of the principled binding together of partial or local information in such a way that these parts and the modes of transit and action supported between such parts are coordinated by virtue of some rule or principle or system into a coherent whole. In the conclusion of this dissertation, I will have a chance to discuss a few other models of continuity to have appeared in the history of mathematics, in relation to which some of the advances of sheaf theory (regarding the formulation of a concept of continuity) will be seen in a larger context.

The concluding chapter steps back from these special developments, presents a number of "models" or characterizations of continuity in more abstract terms, and discusses some of the corresponding representations of generality to which each such model gives rise. The conclusion presents the first steps of the classification of the concept of continuity referred to in the title, by first naming and describing the different models and then providing some initial organization to these various approaches. Of the many distinct models articulated, discussed, and partially compared with one another, some of the most notable include: Question of Scale (Randomness; Idealization); Relation of Parts (Density; Compositional; Reflexivity; Issue of Distinction; Structural); Closeness; Issue of Size; Passage (Local-Global Passage). Some of the specific ideas on continuity-generality presented in the narrative developed in this dissertation are also better situated within this larger field of models, and some "loose ends" from those chapters are tied together. The dissertation ends with brief discussion of some of the dominant ways (and the reasoning behind those ways) in which it has been argued in the past that continuity or discreteness is “better” (or somehow “to be preferred”) than the alternative.

Chapter 1

Aristotle: The Grundbuch of Continuity

> [B]eing and the one start right out already having genera [ὑπάρχει γὰρ εὐθὺς γένη ἔχον τὸ ὂν καὶ τὸ ἕν], on account of which the genera of sciences [ἐπιστημαι] also follow these.
>
> *Metaphysics*, 1004a4-5

Introduction

Once viewed at the appropriate level of generality, the structural similarities and numerous conceptual overlaps between Aristotle's *Physics* and his *Posterior Analytics* can be shown to reflect not some accidental or unjustified transposition of problems and techniques from one apparently unrelated field of inquiry to another, but to reveal certain deep connections between the characteristic structure and aim of demonstration (and the sciences) and the structure of τόπος. While there are important differences between the enterprises of physics, that has for its domain physical bodies in motion, and demonstration, that has for its domain beings in general with decidable properties, careful consideration of Aristotle's work reveals the existence of certain structural similarities between the two that are more than just analogical. The network of connections that exists between the two hinges on the concept of continuity (συνὲχεια). Once seen through the lens of the concept of continuity—together with the constellation of problematics this concept represents—the *Analytics* and the *Physics* begin to shed light on one another, and thereby to shed light on some of the deep-seated and subtle connections that exist between the study of the structure of motion (and place) and the study of the structure of scientific knowing (and demonstration).

The *Physics* shows that continuity is defined by infinite divisibility only accidentally; over and above this negative characterization, it develops a positive and more demanding account of continuity in terms of what holds a continuum (a continuous object, motion, time) together. For any given continuum, Aristotle gives a general account of how its characteristic unity and suggenicity (on which more below, but basically "belonging to the same genus") are precisely secured by the action of its indivisible limit. Taking this reasoning to the limit, the *Physics* works up to the conclusion that only the cosmos as a whole moves continuously, in the strictest sense, and that it does so only on account of the action of the first and indivisible unmoved mover. In this way, something that is characteristically not in or a part of nature, and is characteristically indivisible and unmoving, becomes the sole guarantor of the continuity of nature as a whole and its motions; the first mover is the first thing that is not itself movable but forms the basis of all motion. In the *Posterior Analytics*, the continuity and necessity of each given demonstration (and the continuity between the propositions out of which they are built and their conclusions) is secured by the existence of immediate, indivisible principles. Taking this reasoning to the limit, the continuity and integrity of the entire enterprise of demonstration (and, so, scientific knowing as a whole) is secured by the existence of a kind of knowing, namely *nous*, that knows the indivisible principles and knows them "immediately," and stands as the first kind of knowing that is not a demonstration but which forms the basis of all demonstration.

This chapter starts from the assumption that deeper than any superficial differences between the concerns of the *Physics* and the *Posterior Analytics*, the common structure suggested above points to subtle, but lasting connections between the study of nature and the study of demonstration. I can begin to motivate these connections by observing a basic feature of Aristotle's account of the nature of demonstration and the scientific knowing it produces. For Aristotle, what can be known through scientific knowledge (ἐπιστημη)—even the very possibility of scientific knowledge—is revealed by an analysis of the general structure and limitations of the process of demonstration. Scientific knowledge is built from special sorts of syllogisms, namely demonstrations, or deductions that entail the necessary truth of their conclusions and, in doing so, reveal the grounds for that conclusion. It is a basic fact that the construction of syllogisms in general and demonstrations in particular is fundamentally a matter of finding the "middle" (μεσον) by means of which something can be proven of something else: syllogistic deduction in general is described as a process of establishing a relation between two terms through a series of middles (μεσα),[1] while what distinguishes the sort of syllogism that is demonstration stems from the special nature and (explanatory) function of this middle term. Scientific knowing in general inquires into (1) a fact, (2) the reason or cause for a fact, (3) whether an object exists, and (4) what a thing is. According to Aristotle,

> When we inquire whether something is or is not a fact or whether an object simply [ἁπλῶς] exists or not, we inquire whether it has a middle [μέσον] or not. And when we further inquire, after knowing that something is a fact or that an object exists (i.e., whether in part or simply), into the why of it or the whatness of it, then we ask 'What is

[1]See Aristotle, *Aristotle's Posterior Analytics*, II.4, 91a14-15.

> the middle?' [τί τὸ μέσον] [...] It follows, then, that in all our inquires we inquire either (a) whether there is a middle [εἰ ἔστι μέσον] or (b) what the middle is [τί ἐστι τὸ μέσον]; for the cause [αἴτιον] is a middle, and in all cases it is this that is sought.[2]

Once put in perspective, the pivotal role given to the "middle" throughout Aristotle's characterization of syllogistic logic and demonstration begins to suggest why, in his "meta-logical" reflections in the *Posterior Analytics* on the general features of demonstration and scientific knowledge, not only is his approach consistently "spatial" or "positional," but he continually filters the discussion through various problematics and concepts having to do with continuity (συνέχεια) and (in)divisibility, problematics that importantly overlap, complement, and refine the discussions of continuity from the *Physics*. In the past, a few commentators have remarked, usually merely in passing, on the existence of a certain parallel between the *Posterior Analytics*'s concerns with the possibility of infinite demonstration and the discussions of Zeno in the *Physics*. But to my knowledge no attempt has been made to more systematically correlate these works (beyond the limited discussion of Zeno) or even to pursue this particular parallel in any detail.

This chapter and the following aim to contribute to such a systematic correlation, and it grounds this aim in a closer consideration of Aristotle's characterization of the concept of *continuity* in the *Physics*, on the one hand, and in the explication of certain important connections between the latter and *generality* or *genos* on the other. Initially, then, the present chapter might be thought of as starting to better situate and understand the otherwise curiously high (and, to certain critics, illicit) degree of "spatial" terminology in Aristotle's

[2] Aristotle, *Posterior Analytics*, 89b39-90a2; 90a5-7.

development of syllogistic logic and discussions of the nature of demonstration. Further, regarding the restricted parallel between the Zeno discussions and the infinite demonstration discussions, this chapter and the next aim to clarify that, while those commentators who have noticed this parallel all appear to agree that *Posterior Analytics* reaches the "opposite" conclusion as that reached by the *Physics*, this is at best only partly true; more seriously, it occludes a much deeper consistency that runs between these two works on the basis of an especially powerful, if subtle, connection that has not yet been appreciated. Such consistency emerges out of an appreciation of the connections between continuity and generality (or *genos*), in particular via a concept that I will call, following Aristotle's own coinage, *suggenicity* (συγγενεία).

Translators and commentators have long appreciated the fact that Aristotle's use of the term *genos* is very flexible, and that certainly on the whole *genos* is used in a positively more expansive sense than the narrow notion of the genus of a species.[3] For the most part, generic distinctions—*genos* deployed in its more expansive sense—are meant to capture any

[3]For instance, McKirahan notes that the reduction of Aristotle's account of genus to the narrower notion of the genus of a species is misguided:

> [S]ince nothing in the theory of demonstrative science requires that subject genera have genus/species structure, and since there is clear evidence that a subject genus contains more than just its subjects, we must concede that Aristotle's conception of subject genera is different in origin, purpose, and nature from his conception of genus/species hierarchies. (McKirahan, *Principles and Proofs*, 62)

However, while there appears to be some consensus on the flexibility and breadth of this concept, in general commentators do not seem to be eager to countenance the idea that Aristotle even has a *principled* or *systematic* theory of genera. For instance, James Bogen claims that "[Aristotle] does not seem to use the term 'genus' consistently to mark kinds on any single level of generality. The taxonomic levels of genera seem to differ with context, but as far as I know, Aristotle has no principled story to tell about whether, to what degree, or how, genera levels are determined by context" (Bogen, "Change and Contrariety in Aristotle," 8). Bogen is exactly right to suggest that the levels of genera differ with context; he goes wrong, as will be shown, in suggesting that either Aristotle "ought to" have fixed a single sense or level of *genos* or that he does not have a "principled story to tell" about how the levels are determined by context.

unification of parts or plural realities into a whole that is "greater than" the unity that belongs to an individual being by virtue of strict numerical unity. At the same time, as we shall see, Aristotle consistently holds that such genera are irreducible, an immediate consequence of which is that distinct generals cannot be reduced either to one another or to individuals. Moreover, in contrast to (the standard story about) Aristotle's pronouncements concerning the "universal" (καθόλου), his uses of the concept of *genos* are more consistently meant to have ontological, and not just predicational, import. Aristotle's use of the more nuanced and flexible concept of *genos* allows him to develop a theory of generality as coming in degrees or levels, in which can be seen the beginnings of something like a "relativized" theory of universality. Support for this will be provided by a reconstruction of his sophisticated account of how the limits of continua at once make the whole of which they are the limits *internally* continuous, while *externally* discretizing them, an account that will importantly feature the notion of suggenicity.

An adequate explanation and defense of this last claim, and its larger significance, involves some subtleties to be explained in the course of this chapter and the next. However, for now, I can highlight the two main features of this account as it bears on the issue of generality. First, genera will emerge as characteristically *incomparable* (or *incommensurable*) and, as such, *discretized with respect to one another* (how this discretization is accomplished is what is of the most interest). Second, there is the important (but overlooked) phenomenon of suggenicity, whereby "internally" or within a given *genos*, natures are "fused together" in such a way that *continuity is the rule.* Understanding the process whereby the discretization of distinct genera is accomplished, and how this is not incompatible with, but even requires, that *within* or *between* the limits of a given *genos*, continuity is the rule, is an important task in its own right. That and how this may be the case, and how the concepts of continuity and generality enter each others' orbits in this connection, will be spelled out in detail in what follows.

The primary aim of the next two chapters, then, is to clarify some of the relations between continuity and generality in Aristotle, and in doing so, to contribute to the clarification of some of the bonds linking the study of place with the study of demonstration. The secondary aim is to contribute to a more nuanced and complete picture of Aristotle's characterization of continuity. The complication and sophistication involved in his often rather technical treatment of the concept in the *Physics*, and the problems out of which it was born, have largely served to hide its more global influence from readers, leaving continuity to act as something of an "invisible hand," exerting a powerful but subtle influence over many aspects of his thought. The comparative inattention to Aristotle's refined account of continuity—even restricting our attention for the moment to his extensive and dynamic account of it in the *Physics*—has meant that basic problems having to do with the very coherence of this account have largely gone unremarked or at least unresolved.[4] To mention just one such

[4]The failure to appreciate the central significance of Books V and VI, where continuity is most explicitly treated, is illustrated by Clarendon's Aristotle's *Physics* series that includes separate editions, with detailed notes and commentaries, for Books I and II, III and IV, and Book VIII. Compare this, however, to Joe Sachs's translation and commentary, where he notes that "Books V-VII of the Physics get less attention than do the other books, and when they are discussed it is sometimes only for the sake of finding contradictions between things said there and in the other books.[...] [But] Book 5 is the clearest structural pointer to the shape and unity of the argument [of the entire Physics]" (Aristotle, *Aristotle's Physics*, 144). However, it is revealing that while Sachs appears to appreciate the importance of, and some of the subtleties involved in, Aristotle's treatment and use of the concept of continuity, he apparently did not think that 'continuity' merited placement in the glossary of concepts—a glossary that includes not only all the usual suspects, but even has a definition of 'dog days' (ὑπο κυνα)!

The strangeness of the general inattention to the ubiquity of continuity in Aristotle's *Physics*—not to speak of the widespread neglect of more important issues touching on its systematic importance—is readily made apparent by the amusing fact that the *word* 'continuity' ('continuous', etc.)—συνεχές (ῆ, ὴς, ής, εῖ, εῖς, etc.)—makes 180 appearances throughout the *Physics*. This number should be compared to the following: there are 22 occurrences of ἐντελεχεία; 31 occurrences of ἐνεργεία; 33 occurrences of δύναμις (ιν)(and even if basic problem: while Aristotle's arguments that continua are not *composed of* indivisibles (from *Physics* VI) are well-known and have been much discussed, it is not clear how this requirement is supposed to work together with his equally (if not more) insistent arguments (themselves seldom discussed) that continua are *made* continuous by their indivisible limits (the characterization that features heavily in Book V and resurfaces throughout the *Physics*). Further, Aristotle can also be found to speak, in one breath, of the continuity and *finiteness* (and also *completion*) of a whole, a connection that emerges as especially pivotal in the *Posterior Analytics*. However, the tendency to focus on only those sections of the *Physics* where Aristotle is discussing, in aporetic form, the problems due to Zeno—in which he claims, provisionally, that continuity is characterized accidentally by infinite divisibility—as well as the more intuitively appealing (from a contemporary perspective) alignment of continuity and infinity, have worked to obscure such important features of his account. There is a surprising number of basic problems and discrepancies concerning Aristotle's account of continuity in its own right; it is one of the aims of Chapter 2 to begin to address some of these.

The present chapter is dedicated to establishing some initial connections between continuity and *genos* as they appear in the *Posterior Analytics*. In Chapter 2, I develop those connections as they appear in the *Physics* (and a few other places), before going on to draw out certain relations between the two.

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we include δυνάμ- (-ει, -ενα, -εις), we add only another 72); 56 occurrences of τέλος (ει, ειος, ειον, ους, etc.). The number of occurrences of (all the variants of) 'form' still does not match that of continuity (there are 151 in total). These numbers may be better appreciated by considering that 'nature'—φύσις (ιν, ει, εως, etc.)—occurs 240 times in the entire text (nearly half of which occurrences can be found in the first two books). If we restrict our attention to books III-VIII, while 'nature' occurs 136 times, 'continuity' can be found 170 times! The widespread neglect of continuity would be far more understandable if continuity were not a technical term for Aristotle. However, it is in fact one of the most refined and carefully crafted concepts in his technical arsenal.

Posterior Analytics

As the *Prior Analytics* makes clear, syllogistic deduction is fundamentally about establishing a relation between terms—specifically relations of "inclusion as in a whole" (as well as "exclusion")—via a chain of sub-deductions from two propositions or premises, where a proposition is composed of two terms and a "copula" taking the form of, e.g., 'holds of all', 'holds of none', etc., where one term is said to be predicated (κατεγοριειν) of the other term. Syllogistic deduction is accordingly said to be a linking of the "extreme" (ἄκρον) of the conclusion via (variously arranged) middle terms of the premises. For instance, if we designate 'holds of all' by 'a', then we might have the following syllogism:

$$\frac{\text{AaC} \qquad \text{CaB}}{\text{AaB}}$$

Here, in this "first figure" syllogism (affectionately known as *B***a***r*b**a***r***a**), the middle term C occurs as subject in the premise on the left and as the predicate in the premise on the right. In the second figure, on the other hand, the middle term would function as predicate in both of the premises; in third figure syllogisms, the middle term would be located in the subject position in both of the premises.[5]

Demonstration is a special sort of syllogistic deduction in that its middle terms are *explanatory* or *causal*. In this way, demonstrations do not merely reveal abstract logical relations between terms and the propositions that they are a part of; a genuine demonstration, for Aristotle, amounts to the display of a fact of nature that is at once an explanation

[5]There are 14 figures in total. In most of the discussion that follows—following Aristotle's arguments in the *Prior Analytics* that each figure can ultimately be reduced (or "perfected") to a syllogism in the first figure, and his arguments in the *Posterior Analytics* that the first figure is the "most scientific of all"—we will confine our attention to the first figure.

of that fact. To have scientific knowledge of a conclusion via a demonstration—a process investigated by the *Posterior Analytics*—is a matter of supplying a demonstration that is an explanatory deduction of a necessary truth, i.e., it is not only a matter of supplying a deductively valid (or sound, for that matter) proof, but a matter of knowing the conclusion through its cause. As Aristotle emphasizes, these causes are always shown through a middle (connected with the minor and major premises in a special way) and the finding of such causes always amounts to the finding of such a middle.

That there are propositions that are held to be immediate (ἄμεσα, literally 'without middle')—a key claim of the *Posterior Analytics*—is equivalent to requiring that in their role as explanatory such special propositions need not themselves make appeal to a further middle. Thus, given the proposition 'A holds of all C', if such a predication is not 'immediate', a new middle between A and C must be interposed, one that explains their connection. If the new middle interposed does not result in an immediate and thus explanatorily basic predication, the same interposition must take place. The main problem Aristotle means to confront in the *Posterior Analytics* is determining whether the looming infinite regress can be blocked; a key component of his answer is the requirement of certain immediate propositions.

The principal aim of the *Posterior Analytics* is to delineate the structure of demonstration and, in so doing, to determine its limits (and thereby the limits of scientific knowledge). The sciences consist of two fundamental objects: demonstrations (or proofs) and principles. While there are many problems and themes addressed in the *Posterior Analytics*, the text can be seen as dedicated to the formulation and resolution of three (as we shall see, related) problems:

1. Can there be infinite demonstrations or are demonstrations necessarily finitary or limited? Aristotle's answer to this—that they are finitary—in part motivates and in part is explained by the existence of indemonstrable (αναποδεικτον) principles (ἀρχαι).
2. When is a science 'one'? Aristotle's response will involve two main ideas. (a) The terms involved in the premises and conclusions of a demonstration must be *suggenic*, i.e., belong to the same *genos* or kind; this will guarantee the necessity of the relationship of consequence in any given demonstration, but it will also lead Aristotle to the claim that there can be no "master science," i.e., no science whose principles would not be "proper" to it but would rather embrace the principles of all the other sciences. (b) Demonstration cannot be "transferred" (μεταβάντα) across distinct genera, i.e., demonstration cannot "cross over" from one genus to another.
3. What are the first principles on which demonstration is based? How does the underlying subject matter or genus (γενος, ὑποκειμενον γενος) of a science relate to its first principles? Aristotle's answer to both questions will rely on the central role given to the "proper" first principles of a science (as opposed to those that are "common" to multiple sciences).

Before discussing these three problems and establishing some more systematic relations between them, I define some terms. I follow Aristotle's consistently spatial or positional terminology, language that becomes especially prominent in the pivotal chapters (I.19-23) devoted to the resolution of problem 1 above. In addition to the decisive role played by "middles" (which it is the job of demonstration to find and use to carry out a demonstration) and propositions that are "without middles" (which form the indemonstrable principles on which any demonstration is ultimately based), Aristotle repeatedly refers to the terms in a predication as forming *intervals* (διαστήματα), references that are usually further accompanied by language of (in)divisibility (both ἀδιαίρετος and ἄτομος).[6] Especially notable is Aristotle's use of the language of intervals in his characterization of the main result of the arguments of I.19-22:

> It is now clear that of demonstrations, too, there must be principles, and that, contrary to the assertion of some, as we stated at the beginning, not all things are demonstrable. For if there are principles, then neither (a) are all statements demonstrable, nor (b) is it possible to proceed to infinity; for either (a) or (b) to be the case is nothing else than for there to be no immediate or indivisible interval [διάστημα ἄμεσον καὶ ἀδιαίρετον], but for all of them to be divisible [ἀλλὰ πάντα διαιρετά]. For to demonstrate AC is to insert a term [or terms between A and C] and not to attach a term [or terms] outside [of A or of C]. Thus if this [insertion] can proceed to infinity, an infinite number of terms could exist between two terms [ἄπειρα μεταξὺ εἶναι μέσα]; but this is impossible, if the predications terminate [ἵστανται] in the upward and in the downward direction.[7]

I will discuss the arguments represented here in more detail below. For now, I simply observe that to say either that (a) there is a demonstration of everything or (b) demonstration proceeds to infinity is *equivalent*, on Aristotle's account, to saying that there exists no immediate or indivisible interval, but rather all intervals in a demonstration are divisible. Another decisive passage (in terms of this interval language) comes in I.23:

> Now when it is to be proved [that A holds of B], we should take something, say C, which is predicable of B primarily; and similarly [if it is

[6]For some instances of the interval language, see Aristotle, *Posterior Analytics*, 82b7-9, 84b15. The same language also consistently appears throughout the *Prior Analytics*, for instance at 26b21, 35a31, 38a4.

[7]Aristotle, *Aristotle's Posterior Analytics*, 84a30-84b1.

> to be proved that] A holds of C. Proceeding always in this manner, we should take no premise or term in the proof which lies outside of A but always close pack [πυκνοῦται] the middle terms [τὸ μέσον] until each [of the intervals] becomes indivisible and one [ἀδιαίρετα γένηται καὶ ἕν]. Now [each interval] is one when it becomes immediate [ἄμεσον], and a unitary proposition without qualification [ἁπλῶς] is an immediate proposition. And just as in other things the principle [ἀρχὴ] is one, but not the same in all cases—in weights it is the mina, in music it is the quarter-tone, elsewhere it is some other thing—similarly the one [τὸ ἓν] in syllogisms is an immediate proposition, and in demonstrations and scientific knowledge it is *nous* [νοῦς].[8]

Building on this language, I present a few definitions (following Aristotle's own usage) of concepts that will play an especially prominent role in what follows:

Definition 1.0.1. A predication or premise—e.g., 'A holds of all B' (AaB, or AB for short)—is *mediate* or *divisible* when there exists some middle (μεσον), say C, by means of which a demonstration of AB (from AC and CB) is possible. A predication or premise DE is *immediate* (ἄμεσον) or *indivisible* (ἀδιαίρετον) when there exists no such middle by means of which it might be demonstrated; thus, such a premise is *indemonstrable.* If DE is indivisible, Aristotle will also speak of D holding of E *atomically* (ἀτόμως).

Definition 1.0.2. A term B is said to be *continuous* with C if the proposition BC is immediate or indivisible. By extension, the terms A, B, C, D, E are continuous if the propositions AB, BC, CD, and DE are each, singly, immediate premises or propositions.

Definition 1.0.3. A *chain (or line) of predication* (συστοιχία) is any sequence of predications. A *continuous line of predication* is any such sequence of predications each of which component proposition is immediate, where this means that the terms are ordered such

[8]Aristotle, *Aristotle's Posterior Analytics*, 84b32-85a2.

that given the extreme term A and the extreme term Z, and beginning from the (immediate) proposition AB_i, each (immediate) proposition B_iB_{i+1} is succeeded by $B_{i+1}B_{i+2}$ or by $B_{i+1}Z$.[9]

It is an important feature of Aristotle's account that propositions are not, in general, convertible, where the terms A and B would be convertible if (confining ourselves to universal affirmative propositions) both AaB and BaA are true. Thus, we might represent the connections between terms with directed line segments or arrows; in doing so, we could illustrate what it would mean for a middle term to be *not continuous* with another term in a demonstration.

- For a predicational chain involving A, B, C, D, and E, we can have C acting as the middle term:

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[dr, dashed] \arrow[ddrr, bend right, "C_{middle}"'] & B \arrow[d, "\psi"] & \\
& C \arrow[r, "\tau"] \arrow[dr, dashed] & D \arrow[d, "\upsilon"] \\
& & E
\end{tikzcd}$$

  Here, C is not continuous with either A or E, the major or minor of the demonstration segment. C is of course a cause (as a middle), but it is not the first. (B is the first for A, and D is the first for E.)

- For a predicational chain involving A, B, C, D, and E, in the second case we have B acting as the middle term:

[9] For some instances of the use of this language of continuity or continuous (συνεχὲς) in the *Posterior Analytics*, see I.29 (87b6), II.10 (94a8). II.12 is dedicated almost entirely to continuity, in which context Aristotle remarks that "These facts should become more evident when a universal discussion of motion is considered" (95b12-13), a remark that has led commentators to speculating that the *Physics* was written after the *Posterior Analytics*. Whether or not that is the case, such remarks practically demand comparison with the *Physics* and its discussions of continuity. Unfortunately, in the interests of space, I have not been able to fit a discussion of this interesting and somewhat complicated chapter.

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[dr, dashed] \arrow[drr, bend right, "B_{middle}"'] & B \arrow[d, "\psi"] \arrow[dr, dashed] \arrow[ddr, bend left] & \\
& C \arrow[r, "\tau"] \arrow[dr, dashed] & D \arrow[d, "\upsilon"] \\
& & E
\end{tikzcd}$$

Here, B is not continuous with E, but B *is* continuous with A (via ϕ), and as such, acts as the first cause of AE. This is, importantly, an "unqualified" demonstration, for the first cause of the conclusion of the demonstration, namely AE, is included.

- Finally, we could also have D acting as the middle term:

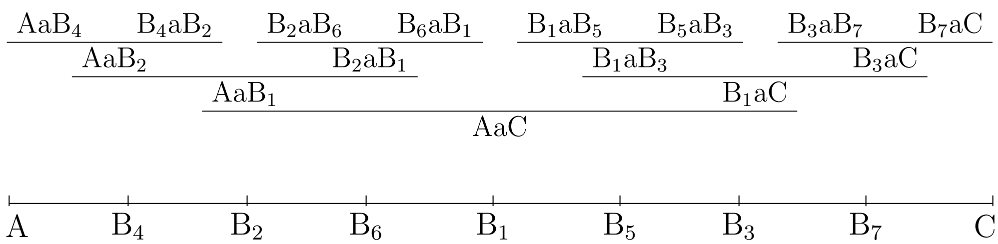
$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddrr, bend right, "D_{middle}"'] & B \arrow[d, "\psi"] \arrow[dr, dashed] & \\
& C \arrow[r, "\tau"] \arrow[dr, dashed] & D \arrow[d, "\upsilon"] \\
& & E
\end{tikzcd}$$

Here, D is not continuous with A, and so the first cause is not given (though it does provide a "distant" cause). But D is continuous with E (and, so, might be its first cause).

One of the purposes of pointing out these distinctions is to observe how we can have a continuous line of predication consisting of, e.g., AB, BC, CD, and DE, without *every* term being continuous with every other term. The definition only requires that the propositions are ordered sequentially in such a way that each successive proposition is continuous with the proposition it follows. Such distinctions do, however, also allow us to perceive an important possibility in the various positions such middle terms can take:

Definition 1.0.4. An immediate premise is *first* if the subject involved in the premise is as general as possible. It will turn out that the "closer" this middle is to the principle, the more general it is. In this way, as demonstrations in the most demanding sense will involve the use of a middle term mediating between two or more premises that moreover acts as a cause of the fact described by the conclusion, demonstrations through the first cause will provide conclusions of maximum generality.

We can consider these definitions in more detail in the context of Aristotle's choice to represent the objects of demonstration as intervals. Consider the proposition AaC. The process of demonstration (via middle terms dividing the interval) could be represented as follows:



The Interval AC Subdivided

The above representation of the demonstration of AaC from seven middle terms would suggest, assuming it is complete, that the propositions at the top, i.e., AaB_4 through B_7aC, are themselves indemonstrable (for they have no middles by construction), and so they would be among the principles of the science deploying such a demonstration. Each of the propositions at the top would thus represent an indivisible or atomic interval. On the other hand, the remaining intervals (AaC in particular) are divisible precisely in being demonstrable. The "close packing" of the interval AaC with new middle terms and new sub-intervals represents, at each instant, a division or 'cut' of the interval AaC. But this is a process, Aristotle will argue, that the demonstrator must carry out until each of the branches terminates in a proposition that is itself indemonstrable or indivisible, i.e., that cannot be divided into other propositions. For Aristotle, just as in music the smallest interval of which larger intervals are multiples acts as an indivisible unit or 'one' and ruling principle of musical intervals and harmonies, in demonstration such indemonstrable premises will act as the indivisible units and ruling principles of a demonstration. Aristotle will later state that such indemonstrable propositions are the elements (στοιχεῖα) of the conclusion, and that these elements are "as many as the middles" (84b20).[10] As such, a conclusion can be regarded as a σύνθετον.[11]

Putting these concepts to work over the next two sub-sections, it will be shown that

1. The structure of demonstration, necessarily involving appropriate middles, requires the existence of indemonstrable (or immediate or indivisible) principles. As indemonstrable (immediate, indivisible), these principles serve as *limits*, revealing the ultimate nature of the underlying *genos* or subject matter of its science—what that science is *about* (περὶ ὃ, 76b22), which is unique for each science (87a38)—and marking off its scope from that of sciences that treat of other genera. In this manner, such immediate principles will serve to *externally*—i.e., from the perspective of other sciences with their own defining genera—discretize a science (together with its underlying *genos*). This is sometimes referred to as Aristotle's notion of the *irreducibility* of distinct sciences. However, it will also be seen how the

[10]As Malink, *Aristotle on Principles as Elements*, 26 correctly observes, Aristotle is off by one. As one can see from the above diagram, the number of immediate propositions in a demonstration will be one more than the number of its middle terms (as is also immediately obvious from the basic fact of syllogisms that a single middle term acts as the middle for *two* premises). But, as Malink further notes, Aristotle does qualify his statement by saying that the elements are "either all immediate premises or the universal ones" (84b22), which might be referring to those immediate propositions not involving the minor term; if this were viable, then his claim that the number of principles as elements is the same as the number of middle terms could be salvaged.

[11]See Aristotle, *Metaphysics*, 1014a26, for instance, for Aristotle's notion that anything composed of its elements forms a σύνθετον.

principles equally form—again, precisely as indivisible limits—the basis of continuous lines of predication and ultimately of the unity of the underlying *genos* of the science itself. In this manner, such immediate principles can be thought of as *internally* securing the continuity of a given science (and its respective subject matter or *genos*). Moreover, it is the internal continuity of given genera that will ground Aristotle's treatment of levels of universality (or generality) as determined by the "nearness" of a middle to the proper principles of that science. This consistently "spatialized" account motivates Aristotle's treatment of universality or generality as *relativized* or *contextualized*.

2. The unity of a science ultimately stems from the unity of its underlying (unique) *genos*. As Aristotle makes clear in various places, "We think we know something [scientifically] if we possess a deduction from some true and primitive premises—but this is not so: the conclusion must be suggenic (συγγενη) with them [the basic principles]."[12] A demonstration consists of three things: (i) essential attributes (πάθη) being demonstrated to belong to some *genos*; (ii) the axioms from which the demonstration proceeds; and (iii) the *genos* which is the "subject" whose attributes are made known by a demonstration.[13] Unlike the "common" axioms, which can be the same across various sciences, there are principles for each *genos* that are "proper to each science,"[14] and the proper principles of that science

[12]Aristotle, *Posterior Analytics*, 76a26-30, translation modified.

[13]Aristotle, *Posterior Analytics*, 75a39-75b2.

[14]Aristotle, *Posterior Analytics*, 76a38.

will include definitions and existence claims regarding the underlying *genos*. The unity of a particular science is secured "internally" by the suggenicity of its proper principles and conclusions; however, "externally," the limits of each *genos*, given by the immediate proper principles of a science, secure the irreducibility of distinct sciences, something that leads to the "no genus crossing" prohibition. The defining limits of these genera act not only to make each science the individual science that it is, but reveal the principle by means of which that of which the science is about is what it is. Just as the power of inductive reasoning, whereby we are given some ability to predict the future, stems ultimately from the predictability and large-scale stability of nature itself—i.e., the fact that for the most part the future, if not determined by, at least resembles antecedent conditions—Aristotle ultimately looks to the structure of nature to account for the most fundamental aspects of the structure of (deductive) demonstration.[15] This suggenicity requirement will thus be further connected to essential predication and the nature of definition.

On Infinite Demonstration

Already as early as *Posterior Analytics* I.2, Aristotle proposes the existence of indemonstrable principles and claims that they are what all science is based on; these principles are initially said to be (in their own right) (1) true, (2) primary, (3) immediate; and (with respect to the conclusions) (4) better known than, (5) prior to,[16] and (6) explanatory of (αιτια), or

[15]This, perhaps above all, is what justifies our attempt to connect these matters with parts of the *Physics*. I discuss what is meant by "looking to the structure of nature" below.

[16]For Aristotle,

causes of, the conclusion.[17] Still more, Aristotle will aim to show that the premises must also be necessary, known in themselves or *per se* (κατ' ἁυτο), and universal.[18] Without such characteristics, there can still be syllogistic deduction; however, to have a *demonstrative* syllogism, we require them. The principles are later revealed to be of three sorts: axioms (sometimes called the "common" (κοινα) principles, since they may apply to more than one science); definitions (of the subjects and attributes pertaining to the particular genus of a science, the meaning of which is assumed); assertions of existence (namely, of "the genus whose essential attributes [a science] examines" (76b)).[19] Having scientific knowledge of things that can be demonstrated is the same as having a demonstration of those things, so if these principles were themselves demonstrable, then knowing them scientifically would amount to having a demonstration of them. But, by definition, proving them would depend on further principles which, again, would themselves require proof. It is precisely for the purpose of evading the infinite regress involved here—one that Aristotle believes destroys

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> Things may be prior and better known in two ways: for what is prior in nature [τῆι φύσει] and what is prior in relation to us [πρὸς ἡμᾶς] are not the same, nor what is better known and what is better known to us. I call prior and better known in relation to us the things that are nearer to perception, and prior and better known without qualification [ἁπλῶς] the things that are farther. The things that are farthest [from perception] are those that are most universal [καθὸλου μάλιστα] and those that are nearest are the particulars. (Aristotle, *Posterior Analytics*, 71b33-72a5)

As he goes on to clarify, when he characterizes the principles as "prior to" and "better known than" the conclusions, he is speaking of what is prior and better known *in nature* or without qualification.

[17] See ibid., 71b20-22.

[18] For a more thorough account of these requirements, see McKirahan, *Principles and Proofs*; for a compact presentation, see Lee, *Science, the Singular, and the Question of Theology*, 8-10.

[19] As McKirahan notes (ibid.), since conclusions can themselves come to be premises, these characteristics should not be seen as a general requirement for *premises*; the point is only that every conclusion ultimately depends on such principles.

the very possibility of scientific knowing—that Aristotle is led to posit that demonstration depends on principles and that, by their nature, such principles are indemonstrable.

Demonstration's dependence on principles that are primary (see (2) above) is further explained as a dependence on "appropriate principles [αρχαι οικειαι]."[20] Whatever is primary in this sense is, moreover, an immediate proposition.[21] The determination of such principles as "appropriate" anticipates a feature that will figure heavily in the rest of Aristotle's account: namely that, via the "proper" (ἴδια) principles of a science—principles that are at once maximally universal and also relative to a given underlying genus—meaning can be given to the idea that generals are relative. In terms of the three sorts of principles mentioned above, the proper principles (opposed to the common axioms) will involve definitions and existence assertions. All scientific knowledge will depend on knowledge of principles. But each particular science will have its own proper principles.

In I.3, Aristotle first mentions the two rival accounts to which he can be seen as responding with his notion of indemonstrable principles. Both rival accounts share the view that demonstration is the only way of producing *episteme*; however, on Aristotle's account, each ends up undermining the very possibility of science. The first group—call them science skeptics—is willing to concede the existence of indemonstrable principles but holds that, on account of the requirement that we must know such principles (and know them *better* than what follows from them), there can be no scientific knowledge in principle (for they assume that *knowing* amounts to demonstrating). The second group—call them science totalists—

[20]Aristotle, *Posterior Analytics*, 72a5-6.

[21]See ibid., 72a6-7.

holds that there is scientific knowledge, but believes, on the basis of the alleged possibility of "circular or reciprocal proofs," that there can be demonstrations of *all* the propositions of a science. In more detail:

- The **science skeptics** believe that either (a) there do not exist explanatorily ultimate principles, or (b) some such principles do exist. Assuming (a), then for each conclusion there will be certain distinct truths explanatorily prior to that conclusion, for which there will be still further distinct truths explanatorily prior to those, etc., leading to an infinite regress, i.e., they must hold that carrying out any given derivation may in principle involve performing infinitely many sub-derivations. Just as Zeno would argue—on the basis of the perceived need to perform infinitely many sub-changes in any change from one state or position to another—that change is impossible, this group is led to hold that the "space" of demonstration is infinitely divisible, on which basis they end up undermining the possibility of science. On the other hand, assuming (b) introduces a problem of its own: if a demonstration produces *episteme*, and given that they further assume that the basic principles grounding the conclusion in question must be objects of *episteme* themselves, to say that they should be objects of *episteme* is just to demand that they be demonstrated. But this leads either to another infinite regress or, assuming Aristotle's characteristics indeed belong to the principles—in particular, that there could be no truths prior to these principles—a demonstration of them would be impossible in principle, and so they would be unknowable.
- The **science totalists** agree with the skeptics in restricting knowing to what can be demonstrated, yet they believe that nothing prevents all statements from

> being demonstrable, on account of their further belief that a demonstration may be circular or reciprocal.

Aristotle states that neither view is correct and that

> Our own teaching is that not all knowledge is demonstrative: on the contrary, knowledge of the immediate premisses is independent of demonstration. The necessity of this is clear; for since we must know the prior premisses from which the demonstration is drawn, and since the regress must end in immediate truths, those truths must be indemonstrable.[22]

While Aristotle offers some initial reasons to dismiss the science totalists, in what remains of I.3, by making a case against the possibility of circular demonstration, at this point Aristotle does not bother to defend his counter-assertion that there are indemonstrable (yet still knowable) principles. His strongest and most direct response (to both groups) is deferred to I.19-22, where he famously argues that it is not possible for demonstration to continue to infinity—or, equivalently, that every demonstration will terminate (ἵσταναι) in finitely many steps, and will do so precisely in being bound "in both directions" by immediate principles. His response requires that he show both that (i) every demonstrable truth has a basis in a finite collection of principles; and (ii) every demonstrable truth can be demonstrated from principles in a finite number of steps. However, on account of the need to defend certain assumptions used without defense in I.19-22, his concern with responding to these two rivals really extends throughout the entirety of the text. Aristotle is thus tasked with drawing together (i) the issue of the finiteness of predicational chains (something he *assumes* in I.19-22) and (ii) the finiteness of demonstration (something he aims to show in I.19-22).

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[22] Aristotle, *Posterior Analytics*, 72b16-23.

The discussion of infinite demonstration begins in I.19 with a preamble on the nature of predication and the structure of deductions composed of such predications:

> If you are making deductions with regard to opinion [ενδοξα] and only dialectically, then plainly you need only inquire whether the deduction proceeds from the most reputable propositions possible; so that even if there is not in truth any middle term for AB but there is thought to be one, anyone who makes a deduction through it has deduced something dialectically. But with regard to truth you must inquire from the basis of what actually holds. There are items which themselves are predicated of something else non-accidentally[...]. [T]here are some items which are predicated of things in themselves [καθ' αὑτὰ)].[23]

The point of this—and other passages that reinforce this point—is to signal to the listener that in dealing with the problem of an infinity of predicational chains and infinite demonstration, he will not be addressing himself to the more or less trivial observation that as humans engaged in the activity of dialectics (which is importantly *not* science) we *could* always expand proofs *indefinitely.* Nor will he be concerned primarily with cases of accidental (κατὰ συμβεβηκός) predication. Rather, in I.19-22 he is concerned primarily with cases of *genuine predication* (ἁπλῶς) and whether nature or reality supports infinite chains of essential predication.[24] This is an important clarification, for among other things, it indicates that Aristotle is *not* concerned with the possibility of *potentially infinite* demonstrations in I.19-22, but rather takes himself to be considering the counterfactual possibility that such infinite demonstrations (actually) exist.

[23]Aristotle, *Posterior Analytics*, 81b19-23...b29.

[24]See also 83a19-23 for another clear statement that he is here concerned with genuine, not accidental, predication.

It is immediately following this important preamble that Aristotle offers his statement of the (threefold) problem of infinite demonstration:

> Let Γ be such that it does not further hold of anything else [ἄλλῳ] and let B hold of it primitively (that is, there is no distinct intermediate between them). Again, let E hold of Z in the same way [i.e., primatively], and Z of B. Now [1] must this come to a stop, or is it possible for it to proceed to infinity [εἰς ἄπειρον ἰέναι]?
>
> Again, if nothing is predicated of A per se [καθ' αὑτὸ], and A holds of Θ primitively and between them there is no prior intermediate, and Θ holds of H, and this [H] of B, [2] must this come to a stop, or is it possible for this to proceed to infinity? This [i.e., [2]] differs from the earlier question [i.e., [1]] to this extent: that question [i.e., [1]] asked if it is possible to begin from something which holds of nothing else while something else holds of it and to go on upwards to infinity; whereas the second question has us begin from something which is predicated of something else while nothing is predicated of it and inquire if it is possible to go downwards to infinity.
>
> There is also this question [3]: if the extremes [τῶν ἄκρων] are fixed, is it possible for the intermediates [τὰ μεταξὺ] to be infinite? I mean, e.g., if A holds of Γ, and B is a middle [μέσον] for them, and for A and B there are other middles, and between these there are still others, is it possible for the middle terms between A and Γ to proceed to infinity or is this impossible?
>
> Now this is the same as to inquire (a) whether demonstrations can proceed to infinity and if there is demonstration of all things, or (b) whether between [any two terms] there is a limit [πρὸς ἄλληλα περαίνεται].[25]

There are three main concerns here (corresponding to the numbers above):

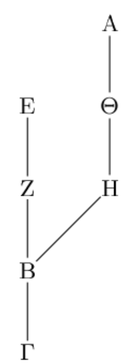
1. Infinitely 'ascending' (more and more general) chains of predication: starting from a fixed subject—more specifically from a subject that is the subject of a predication and that cannot itself be predicated of another subject—can there be an infinite chain of predication, never terminating in an ultimate predicate?

[25] Translation modified from Aristotle, *Aristotle's Posterior Analytics*; and Aristotle, *Posterior Analytics*, 81b30-82a9.

2. Infinitely ‘descending’ (more and more particular) chains of predication: starting from a fixed predicate—more specifically from a predicate that fixes immediately to something and is not itself the subject of any predication—can there be an infinite chain of predication, never terminating in an ultimate subject?

3. Infinite middles: fixing the extreme terms, can there be an infinite number of middle terms between them, i.e., can a necessary connection between a subject and a predicate, as revealed through a demonstration, pass through infinitely many explanatory middles?

Crager observes that Aristotle’s choice of letters in the passage above indicates that in formulating the first two questions, he must have had in mind a single diagram (one that joins the two problems by the repetition of B):[26]



In terms of this diagram, then, the first question asks whether—from a fixed ‘minimum’ Γ to which B is predicated immediately—we can ‘ascend’ infinitely ‘atop’ this B, or whether there must exist an X to which nothing else belongs immediately (but that belongs to each of the other terms all the way down to B). In terms of the interval terminology, we

[26]Crager, “The infinite in Aristotle’s logical epistemology,” 33n.

could phrase the first question thus: given a fixed rightmost endpoint B (that immediately attaches to Γ), must there be a left endpoint, or does the interval extend 'leftwards' out to infinity? The second question, on the other hand, asks whether—from a fixed 'maximum' A predicated immediately of some Θ—we can proceed to infinity predicating Θ of some Θ_1 which is then predicated of some Θ_2 which is then, etc., not reaching in a finite number of steps some Θ_n immediately predicated of H, or whether we must reach such a Θ_n which is predicated immediately of H which is predicated immediately of B which in turn is predicated immediately of Γ. In terms of the 'interval' terminology: given a fixed left endpoint A immediately attached to Θ, if this is bound on the right by the immediate connection B-−Γ, in trying to reach B, must we subdivide the interval between Θ and B to infinity, or must there be a finite number of immediate connections joining Θ ultimately to B? The idea here is that, beginning with A, we want to know if 'above' some fixed minimal immediate connection B−Γ, there can be an infinite 'downwards' chain.[27]

Aristotle proceeds by reducing the third question—whether there can be an (actually!) infinite collection of middle terms between two fixed extremes—to the first two; more specifically, he argues that a negative answer to both the first and the second questions suffices to yield a negative answer to the third question:

> It is clear that if the predications terminate in both the upward and the downward direction (by 'upward' I mean the ascent to the more universal [μᾶλλον καθόλου], by 'downward' the descent to the more particular [τὸ κατὰ μέρος]), the middle terms [τὰ μεταξὺ] cannot be infinite in number. For suppose that A is predicated of Z, and that the middles—call them M M′ M″...—are infinite, then clearly one might start also from

[27]As Crager notes (ibid.), this is the point of insisting on the letters used by Aristotle himself: for Aristotle is considering precisely the possibility of predicational "down-chains" in structures with minimal elements.

> A and find one term predicated of another in the downward direction to infinity (since there will be infinite middles [ἄπειρα τὰ μεταξύ] before arriving at Z); and equally, the middle terms from Z in the upward direction will be infinite before A is reached. It follows that if these processes in the upward and downward direction are impossible, then there cannot be an infinity of middles between A and Z. Nor does it make any difference in the result to urge that some terms of the series AMZ are contiguous [ἐχόμενα] with each other and so exclude other middles, while others cannot be so taken; for whichever terms of the series M you start with, the other middle terms, whether up to A or down to Z, must be either infinite or not infinite. But wherever the infinite series begins, whether from the first term [πρῶτον] or from a later one, does not make a difference; for the terms following after them will in any case be infinite.[28]

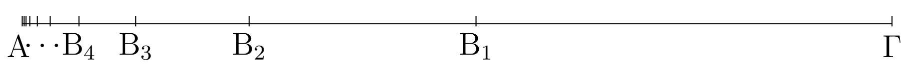
The basic idea of the argument is to show that a demonstration could continue to infinity in either direction ('upwards' or 'downwards') only if there exists an infinite predicational chain, something that I.22 argues cannot be the case. I.21 takes up demonstrations of negative conclusions and claims that essential predications must terminate, assuming they terminate in both directions in demonstrations of affirmative conclusions; this chapter argues by cases, treating syllogisms with a negative conclusion in every figure. We do not need to consider these arguments in detail; it suffices to report Aristotle's basic strategy and conclusion. Aristotle argues that a genuinely infinite demonstration would require the existence of an infinite chain of predication between the terms of the conclusion. The idea can be phrased, following Lear's analysis,[29] in terms of König's lemma, according to which any finitely-branching tree can have infinitely many nodes only if there is at least one infinite branch in the

[28]Aristotle, *Aristotle's Posterior Analytics*, 82a21-a36.

[29]See Lear, *Aristotle and Logical Theory*.

tree.[30] Thus, assuming chains of predication leading to an affirmative conclusion are finite—i.e., assuming the answer to both (1) and (2) is negative—it follows that a demonstration with, e.g., a universal affirmative conclusion can only consist of finitely many steps. For if the premises used in the demonstration are themselves demonstrable, one simply continues the process—in the present case, demonstrating them in *Barbara*. There can be no infinite demonstration in *Barbara*, though, for this would require an infinite chain of predication, so this process must terminate after finitely many steps in immediate principles.

One way of beginning to understand the overall argument is to consider more closely how question (2) is different from (3). In terms of the 'interval' terminology, (3) asks whether—given some fixed maximal predicate A and some fixed minimal predicate Γ, and given that A is predicated of Γ—there can be an infinite number of middles. In other words, given fixed left and right endpoints of an 'interval' (AΓ), can this interval be (actually) subdivided to infinity, e.g., for middle terms B_i, B_j are there always middle terms B_k such that $B_i < B_k < B_j$ (where $B_i < B_j$ whenever B_iB_j but not B_jB_i)? (2), on the other hand, asks whether—given some fixed maximal predicate A and some fixed minimal predicate Γ—one can subdivide the interval to infinity as indicated in the following:



The Interval AΓ Subdivided

Aristotle told us that question (3) amounts to asking (a) whether there are infinite demonstrations (and so demonstrations of everything), or (b) whether between any two fixed terms

[30]Whatever else one wants to say of Lear's analysis, this does appear to be more or less what Aristotle has in mind when he says "Nor does it make any difference...", i.e., it does not make a difference if we assume that one (or more) of the branches has closed; to get an infinite chain of predication, we need just *one* branch that does not close.

there is a limit. If the 'interval' were infinitely divisible, this would yield an infinite demonstration (since it would require the existence of an infinite predicational chain). If, on the other hand, it is not infinitely divisible, this entails that in dividing the interval AΓ, *every* branch would ultimately terminate and the resulting two successive and immediate propositions in which it terminates would necessarily 'meet' in a middle term that does not itself admit further division into a middle term. This amounts to saying that the two propositions to each 'side' of the final division *share a limit.* Say this middle is B_n and the two propositions are AB_n and $\mathrm{B}_n\mathrm{B}_{n-2}$. That neither interval can be further subdivided means that both AB_n and $\mathrm{B}_n\mathrm{B}_{n-2}$ are immediate propositions and their terms are continuous. Together, they form a continuous line of predication from A to B_{n-2}. That they meet in such a middle term in a finite number of steps follows from the negative answer to (1), and by the negative answer to (2) we can be sure that it will take only finitely many steps to get from this immediate proposition AB_n back to the proposition $\mathrm{B}_1\Gamma$. The negative answer to problem (2) thus assures us that the passage 'down' from AB_n and $\mathrm{B}_n\mathrm{B}_{n-2}$ to $\mathrm{B}_1\Gamma$ passes through only finitely many propositions—thus ultimately yielding a finite demonstration of AΓ. As we will see, this entails that the resulting (finite) collection of propositions used to demonstrate the conclusion AΓ can be written as a continuous line of predication composed of a minimal collection of ordered immediate propositions.

Thus, the idea is that if the division did not terminate, there would have to be some branch of the tree that never terminated in an immediate proposition. The answer to problem (1), which Aristotle presently assumes, assures us that this cannot happen. So the tree terminates in every branch. Since the tree terminates 'upward' in every branch, we can select the ‘highest’ branch.[31] We first highlight each of the individual (of necessity, immediate) propositions on the(se) highest branch(es). We illustrate this with the following example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\colorbox{yellow}{$\mathrm{AaB_6}$} \quad \colorbox{yellow}{$\mathrm{B_6aB_4}$}}{\mathrm{AaB_4}} \quad \mathrm{B_4aB_2}}{\mathrm{AaB_2}} \quad \mathrm{B_2aB_1}}{\mathrm{AaB_1}} \qquad \dfrac{\dfrac{\mathrm{B_1aB_5} \quad \mathrm{B_5aB_3}}{\mathrm{B_1aB_3}} \quad \mathrm{B_3a\Gamma}}{\mathrm{B_1a\Gamma}}}{\mathrm{Aa\Gamma}}$$

By construction, each of these highlighted propositions is immediate. But so too are those topmost ‘leaves’ of the remaining branches of the tree with height less than the given highest branch(es):

$$\dfrac{\dfrac{\dfrac{\dfrac{\colorbox{yellow}{$\mathrm{AaB_6}$} \quad \colorbox{yellow}{$\mathrm{B_6aB_4}$}}{\mathrm{AaB_4}} \quad \colorbox{yellow}{$\mathrm{B_4aB_2}$}}{\mathrm{AaB_2}} \quad \mathrm{B_2aB_1}}{\mathrm{AaB_1}} \qquad \dfrac{\dfrac{\colorbox{yellow}{$\mathrm{B_1aB_5}$} \quad \colorbox{yellow}{$\mathrm{B_5aB_3}$}}{\mathrm{B_1aB_3}} \quad \mathrm{B_3a\Gamma}}{\mathrm{B_1a\Gamma}}}{\mathrm{Aa\Gamma}}$$

We continue ‘down the tree’ with this process until there are no further branches that need to be ‘painted’:

$$\dfrac{\dfrac{\dfrac{\dfrac{\colorbox{yellow}{$\mathrm{AaB_6}$} \quad \colorbox{yellow}{$\mathrm{B_6aB_4}$}}{\mathrm{AaB_4}} \quad \colorbox{yellow}{$\mathrm{B_4aB_2}$}}{\mathrm{AaB_2}} \quad \colorbox{yellow}{$\mathrm{B_2aB_1}$}}{\mathrm{AaB_1}} \qquad \dfrac{\dfrac{\colorbox{yellow}{$\mathrm{B_1aB_5}$} \quad \colorbox{yellow}{$\mathrm{B_5aB_3}$}}{\mathrm{B_1aB_3}} \quad \colorbox{yellow}{$\mathrm{B_3a\Gamma}$}}{\mathrm{B_1a\Gamma}}}{\mathrm{Aa\Gamma}}$$

From this small example, the general algorithm should be rather obvious. Intuitively, what the algorithm guarantees is that, after painting in this way, if you were to look at the entire interval AΓ with all its subdivisions ‘from above’, you would see the interval colored (1)

[31] If there are multiple branches with the same height, then we select all of them.

without gaps and (2) uniformly (never, at any part, with more than one coat 'underneath'—except, notably, at the shared endpoints, which gives us the number of "elements"). What the 'painting' indicates is the selection of all of the propositions in the demonstration that, individually, are without a middle. Note that all of those propositions left unpainted are precisely those propositions with a further middle, i.e., capable of demonstration. By definition, then, for each of the propositions picked out by the painting, the two terms involved will be continuous. Moreover, taking all of these propositions (no more and no less) together, with their induced ordering, gives a continuous line of predication. In our example above, AaB_6, B_6aB_4, B_4aB_2, B_2aB_1, B_1aB_5, B_5aB_3, and $B_3a\Gamma$ forms a continuous line of predication beginning with A and ending with Γ. Such continuous lines of predication involving only immediate propositions always give the minimal propositions sufficient for securing the conclusion, i.e., for 'recomposing' the interval.

The reader who is familiar with Aristotle's discussions of divisibility in the *Physics* and the standard interpretations thereof may find this choice of terminology rather strange. But, as we shall see when we come to the *Physics*, this use of the concept of continuity is entirely in line with the most general account of continuity Aristotle provides in that text. Moreover, even without such an analysis, on closer consideration, this choice of terminology is entirely natural. For consider that whenever all the tree's branches terminate—as Aristotle argues they must—there is an algorithm for 'painting' the entire tree or interval without gaps and uniformly. But consider what would happen if one (or more) of the branches never terminated, i.e., if demonstration were infinite on account of one sub-interval being infinitely divisible. Then, there would be at least one branch that, in terms of the algorithm, we could never 'paint'. But the existence of such a branch would be nothing other than the elimination of one of the propositions from the minimal collection of ordered propositions responsible for ensuring that the conclusion can be reached, resulting in a 'gap' in the demonstration of the interval AaΓ. As far as the interval of the conclusion is concerned, then, such a non-terminating ('un-paintable') branch would represent a 'cut' or 'hole' in the sequence of propositions necessary for securing the conclusion. There would simply be no way of getting from one 'side' of the non-terminating branch to the other.

It is indeed quite natural to think of such a 'cut' as a *discontinuity* in the demonstration; and, as we saw, a demonstration without such a non-terminating branch would necessarily have a minimal collection of propositions each of which, individually, joined its two component terms immediately (and so continuously), and that, collectively, was ordered so that for each pair of successive propositions, they share an extreme term. This latter observation is just to say (as Aristotle himself indicates) that the same extreme term that joins any two successive propositions in the chain is their common *limit*. Stringing these successive propositions together, one by one along their shared limits, yields a continuous chain of prediction. Recalling the main definition of continuity from the *Physics*—namely as that for which the limits at which the extremities touch become one and the same and hold together—it is obvious that this use of continuity is in fact not strange at all, but closely allied both to our more intuitive notions of continuity (as not leaving 'gaps') and to his characterization of the concept in the *Physics* in terms of shared limits.

What *should* seem strange, however, is the fact that Aristotle's entire analysis of demonstration and his argument for its finiteness rests on the existence of certain special propositions determined to be *without middle* or *indivisible*. For that means that the continuity of demonstration rests on precisely the existence of certain indivisibles. We are used to thinking of Aristotelian continuity in precisely the opposite terms: namely, as involving what is divisible and infinitely so. To make this even more poignant: Aristotle claims that "the immediate premises, whether all or those that are universal, are elements."[32] To the extent that demonstrations involve continuous lines of predication and these can be used to recompose the demonstrated interval from its minimal collection of immediate (and so indivisible) principles, this seems to conflict with the standard story—a story I will show to be incomplete—that Aristotle held continuity and infinite divisibility to be one and the same and that a continuum is not "composed of" indivisibles. I will spend some time showing, in the next chapter, how this leaves out an important part of the story, and how infinite divisibility does not represent Aristotle's complete characterization of continuity.

For now, let us step back and take stock. In I.20, Aristotle had argued that blocking the possibility of both (1) infinitely ascending and (2) infinitely descending chains of predication would entail the impossibility of (3) infinite middles. In I.21, he shows how this works in particular cases. So far, following Aristotle, we have simply *assumed* that (1) and (2) could be given a negative answer. The reader is no doubt awaiting Aristotle's proof that (1) and (2) can be answered in the negative. However, Aristotle's arguments for a negative answer to (1) and (2) ultimately depend on the repeatedly invoked assumption: "if genuine predication is finite. . . ." But when Aristotle finally offers, in I.22, some defense of this assumption that chains of predication are finite, he in fact pushes the burden of proof back one level further:

> In the case of predicates constituting the essential nature of a thing, it clearly terminates, seeing that if definition is possible, or in other words, if essential form is knowable, and an infinite series cannot be

[32] Aristotle, *Aristotle's Posterior Analytics*, 84b22-23.

> traversed, then predicates constituting a thing's essential nature must be finite in number.[33]

Thus we observe the more fundamental claim emerge: something whose predicates were infinite, or some nature that could support infinite predicates, would not be *definable*[34]—a claim that really amounts, on Aristotle's account, to a claim about *nature*, namely that such a thing would not have an essence (for all essence must be definite or finite). The fundamental idea here is to first connect the (in)finitude of demonstrations with the (in)finitude of predicational chains, and to show that demonstrations can continue to infinity only if there are infinite predicational chains. In defense of the assumption that chains of predication are finite, Aristotle thus ultimately appeals to the structure of *nature*. As Lear writes,

> The entire argument thus far rests upon the assumption that chains of predication are finite.[...] *Posterior Analytics* A22 tries to prove this assumption. The strategy is to appeal to a structure implicit in nature. Chains of predication are not abstract mathematical entities; they reflect an order possessed by a subject and its predicates. This order is reflected in the structure of a proof and restricts the proof to finite length. A study of nature can therefore reveal an important property of proofs.[35]

Going beyond what Lear himself offers on this point, I can be more explicit about what is meant by this appeal to a "structure implicit in nature." In I.22, Aristotle is dealing with essential predication, and can be interpreted as arguing that no essence is infinitely complex, where this is true because any being that has an essence will have only finitely many items

[33] Aristotle, *Posterior Analytics*, 82b36-83a1.

[34] Ibid., 83b8.

[35] Lear, *Aristotle and Logical Theory*, 30. Given observations of this sort, it is surprising that commentators have not generally bothered to look more closely at the connections between the problems and arguments of *Posterior Analytics* and those from the *Physics* which bear a striking similarity—in terminology, in framing of the problem, in the reasoning involved.

predicated essentially of it, i.e., in its τι εστι, and an argument can be made that this is ultimately equivalent to stipulating that every essence has finitely many parts.[36] Aristotle develops this appeal to the finiteness of essence in the following terms:

[36]Crager, "The infinite in Aristotle's logical epistemology," 64-68, makes a case for taking these to be "ultimately equivalent," i.e., that the possibility of real definition entails the thesis that every essence has finitely many parts. Closely related to this argument, it is worth noting Crager's arguments against the Barnes reading of I.22 (i.e., that every essence is finite *because* every essence supposedly can be comprehended by a finite human intellect): Crager notes that neither in I.22 82b27-83a1 nor in recapitulations of this argument does Aristotle mention human beings; more generally, such texts do not give indications of being specifically concerned with the capacities of creatures with limited intellects. Instead, Crager writes, the passage in question is more fruitfully read in light of Aristotle's conception of the infinite:

> Characterizing what the term 'infinite' means, *Physics* III.4 tells us: x is infinite iff (i) x is the sort of item that can be gone through one thing after another, but (ii) it is impossible to 'traverse' [*dielthein*] all of x. It is important to recall that the impossibility here is impossibility full-stop. For Aristotle, the nature of the infinite is to be endless, and to admit no complete traversal. The infinite as such cannot be fully gone through: neither by us nor by anything else. [...] *Posterior Analytics* I.22 82b27-83a1 really should be read alongside Aristotle's claims in both *Physics* I.4 (187b7-13) and III.6 (207a25-32) that 'the infinite **qua infinite** is unknowable [*agnoston*]'. Linguistically, the two *Physics* texts are remarkably close to *Posterior Analytics* I.22. And in both of those contexts, the idea is clearly that the infinite is unknowable per se, by its own nature, and not just unknowable by us. I see no reason to attribute to Aristotle something different in *Posterior Analytics* I.22 82b27-83a1. (Crager, 66)

I agree with Crager's more general account that these passages are not meant to address our epistemic capacities, but rather concern the "architecture of reality" (Crager, 64). The broader and express interest, especially as outlined in I.19 and I.22, with *genuine predication* (or what commentators on *Posterior Analytics* sometimes call 'natural predication'), means that the propositions involved in the demonstrations of interest "must be genuine predications whose grammatical structure mirrors metaphysical structure" (Crager, 62). Accordingly, the task of I.22 is to argue that reality admits no infinite predicational chains: "[The thesis of I.22] is thus more a claim of metaphysics than logic" (Crager, 62). The point of the notion of 'genuine' or 'natural' predication is not to create two different 'species' of predication—the 'natural' and the 'unnatural'—but to insist on the fact that certain predications, and those that enter into the demonstrations of interest, "reflect the real structures that demonstrative sciences make their demonstrations about" (Crager, 19). In other words, the purpose of such passages is to insist on the fact that even though predicating [*kategoriein*] can be understood as a linguistic action, certain predications, namely those that are 'genuine', amount to affirmations that a being underlies another being as its metaphysical 'subject', that a certain being holds of or belongs to another being—where this is taken to refer to a feature of reality, rather than as a merely linguistic act.

In short, then, the reason for speaking of a "structure of nature" is due to both this concern with general predication and the number of linguistic and conceptual connections between the passages of I.19 and I.22 and the *Physics*'s treatment of questions of infinity and infinite divisibility, leading to the idea that the claim that essence is finite can naturally be understood to be a claim about the '(in)divisibility' of nature. Aristotle can be construed as assuming that everything predicated essentially of a being is part of the essence of that being; anything that is a part of the essence of a being, in the context of genuine predication, is to be taken to refer to actual beings that are the principles of the whole they compose.

> In the demonstration of the impossibility of infinite [ἄπειρα] ascent, every predication displays the subject as somehow qualified or quantified or as characterized under one of the other categories, or else is an element in the substantial nature; these latter are finite in number [πεπέρανται], and the number of the widest kinds [τὰ γένη] under which predications fall is also finite [πεπέρανται], for every predication must exhibit its subject as somehow qualified, quantified, essentially related, acting or undergoing, or in some place or at some time.[37]

Moreover, because all essential attributes "must so inhere in the subject as to be commensurate with the subject and not of wider extent," it follows that "attributes which are essential elements in the nature of their subjects are equally finite: otherwise definition would be impossible."[38] The idea here seems to rely on the assumption that the number of underlying genera, or at least the number of attributes that are "commensurate" with it, is finite. I do not have space in this chapter to consider a defense of this extremely vexed issue, though we will shortly discuss a closely related issue. I simply note that immediately following the previously cited passage, Aristotle makes very explicit the overall structure of the main argument of I.19-22 and the dependencies between the various claims and assumptions:

> Hence, if all the attributes predicated are essential and if these cannot be infinite, the ascending series [of predication] will terminate, and consequently the descending series also. If this is so, it follows that the middles between any two terms are also always finite in number. *An immediately obvious consequence of this is that demonstrations necessarily involve principles*, and that the claim of some—referred to at the beginning—that all truths are demonstrable is wrong.[39]

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[37] Aristotle, *Posterior Analytics*, 83b13-17.

[38] Ibid., 84a23-25;a25-27. The 'nature' referred to in this passage is not identical with the 'nature' in the "structure of nature" discussed above; however, on account of what was said in the penultimate footnote, this 'nature' ultimately refers to a feature of the "structure of nature.".

[39] Ibid., 84a27-33, my emphasis.

Aristotle thus invites us to look for the answer to the original question concerning infinite demonstration in a closer examination of the nature of the (necessarily finite) essential predications wherein an attribute is shown to be commensurate with its subject genus.

Suggenicity

Each science is defined by its "proper principles," i.e., by definitions and existence claims regarding its underlying genus; the basis of such proper principles of a science is thus given by its genus or subject genus (γένος, ὑποκείμενον γένος). Initially, the genus or subject genus can be thought of as determining what a science is *about*, e.g., geometry is about spatial magnitude. A genus in this sense is a network of subjects and attributes, and it is the job of demonstration to reveal the *per se* or non-accidental relations of this network. The general idea that in demonstrative syllogisms all the terms in a proof must be of the same genus or be suggenic is thus something Aristotle is repeatedly occupied with in the *Posterior Analytics*. Early on, in I.7, this concern takes the form of an argument that unless the terms of a demonstration come from the same genus, the demonstration will contain terms that are related *accidentally*, something that disqualifies the constituent propositions from even forming a demonstration (in the more demanding sense he is building up to). Rephrased in the form of a positive result: the requirements that all the terms *do* come from the same genus and that there is no "genus crossing" between distinct sciences or no "transference" (μετάβασις) from a demonstration in one science to another,[40] ensures that

[40]This latter claim is qualified by the two (telling) exceptions: (1) when dealing with the axioms or common principles that apply to multiple genera, and (2) when dealing with a science that is subordinated to another (in which cases the genera are said to be the same "in a way" (75b9)).

the terms of the demonstration are related, not accidentally, but *per se*. The key passage regarding the impossibility of transference of proofs across sciences reads:

> It is not possible to demonstrate by crossing from another genus [ἐκ ἄλλου γένους μεταβἀντα]; for instance, [it is not possible to demonstrate] what is geometrical by means of arithmetic.[. . .] But arithmetical proof always has the genus with which the demonstration is concerned [ἀεὶ ἔχει τὸ γένος περὶ ὃ], and similarly for the others. And so the genus must be either the same without qualification or somehow the same if the demonstration is going to cross. Otherwise it is clearly impossible, for the extremes [τὰ ἄκρα] and the middles [τὰ μέσα] must be from the same genus [τοῦ αὐτοῦ γένους]. For if they are not per se [καθ᾽ αὑτά], they will be accidents [συμβεβηκότα].[41]

The main idea here can be isolated as follows:

Proposition 1.0.0.1. (*No Genus Crossing*) Demonstration, if it is to be a demonstration of *per se* relations, cannot cross over genera, on account of the following two facts:

- (i) Each science must always have all of the terms in the premises and conclusions of its demonstrations belonging to the same genus; if they did not, then the terms could only be related accidentally, and I.6 showed that if the terms are related accidentally, rather than per se, then it will not be a demonstration. In short: demonstration concerns terms related per se, and terms related per se must be in the same genus.
- (ii) Each science is individuated by the distinct genus with which it is concerned.

I.9 introduces another perspective on this question, beginning with the following observation:

> Since it is clear that you cannot demonstrate anything except from its own principles [ἐκ τῶν ἑκάστου ἀρχῶν] if what is being proved is to hold

[41] Aristotle, *Posterior Analytics*, 75a38-39...75b7-12.

> of it as such [ὑπάρχῃ ᾗ ἐκεῖνο], scientific knowledge [ἐπίστασθαι] is not simply a matter of proving something from what is true and indemonstrable and immediate. Otherwise it will be possible to prove things in the way in which Bryson proved the squaring of the circle. Such arguments prove by taking as their middle a common feature [κατὰ κοινόν]—a feature which may also hold of something else—and consequently they would apply [ἐφαρμόττουσιν] equally to subjects different in genus [οὐ συγγενῶν]. They thus provide knowledge of an attribute only as inhering accidentally [κατὰ συμβεβηκός], not as belonging to the subject as such—otherwise the demonstration would not have applied [ἐφήρμοττεν] to another genus [ἄλλο γένος] as well.[42]

Here Aristotle begins to address an issue that grows in importance throughout the first book of *Posterior Analytics*: that demonstration in the strictest sense (revealing per se relations) cannot proceed from the "common" principles but must be from principles "proper to" the genus in question. He goes on to add that our knowledge is non-accidental precisely when

> (a) we know that [an attribute] belongs [to the subject] in virtue of [καθ'] that [subject], and (b) that it belongs to it from the principles of that subject as that subject, e.g., when we know that the equality of the interior angles [of a triangle] to two right angles belongs to a subject in virtue of that subject and from the principles of that subject. And so if C belongs to A essentially [καθ' αὑτὸ], it is necessary for the middle term B to be suggenic [συγγενεία] [with them].[43]

Proposition 1.0.0.2. (*Suggenicity of Essential Predications*) Demonstration of essential, non-accidental, attributes of a subject is possible only if the middle terms are suggenic with the extreme terms.[44]

Aristotle remarks on another important consequence of latter claim: that it is clear that the proper principles of a science cannot be demonstrated; for if they could be demonstrated, they

[42]Aristotle, *Posterior Analytics*, 75b38-76a3, translation modified.

[43]Aristotle, *Aristotle's Posterior Analytics*, 76a5-10, translation modified.

[44]This was just (i) above, responsible for the *No Genus Crossing* claim of Proposition 1.0.0.1.

would have to be demonstrated on the basis of other principles, principles that, in the limit, could only be "principles of all, and the science having those principles would be fundamental [κυρία] to all [the other sciences]."[45] Aristotle is here considering the possibility that all the proper principles of all conclusions of all sciences might be provable from a further collection of ultimate principles, i.e., he is entertaining the possibility of a "master science," a science capable of demonstrating the principles of all the other sciences. But the suggestion that there might exist such a "master science" that would have for its principles such ultimate principles is rejected precisely on account of the fact that there is no genus crossing.[46] If a science could exist that would prove the proper principles of different sciences, it would have to be able to prove conclusions concerning distinct genera, and so the conclusions would not be suggenic with the principles, which we know cannot happen on account of the *Suggenicity of Essential Predication*. Aristotle observes that it is precisely on account of the resulting ban on transference across genera that

> it is difficult to know whether you know something or not. For it is difficult to know whether or not our knowledge of something proceeds from its proper principles—and this is what it is to know something. We think we know something if we possess a syllogism from some true and primitive premisses. But this is not so: the conclusion must be suggenic [συγγενῆ] with the first principles.[47]

There are many passages to this effect: it is not enough for the principles of a demonstration to be true, immediate, and primitive; to have a genuine demonstration, the conclusion and the first principles must be suggenic—a requirement that further appears at the level of some

[45]Aristotle, *Aristotle's Posterior Analytics*, 76a18-19.

[46]See Aristotle, *Posterior Analytics*, 76a22-23.

[47]Ibid., 76a26-30, translation modified.

Z belonging *essentially* to X, in which case the middle term Y must be suggenic with X,[48] as well as "negatively," such as when it is said that "the principles of things that differ in genus are different in genus."[49] In particular, this suggenicity requirement will disqualify the "common" principles or axioms from serving as the principles for demonstration in the most demanding sense.

However, in blocking the possibility of a master science and of the existence of "principles of all," Aristotle does not at all mean to block the possibility of universality. In fact, in Aristotle's discussion of universality and particularity in I.24, where he argues from multiple perspectives for the "superiority" of universal demonstrations over those that are particular, he argues that "to prove something more universally is to prove it through a middle term which is nearer the principle," and ultimately "that which is nearest the principle is immediate, and this is a principle."[50] As we know, to have a genuine demonstration of per se relations, the principle and the conclusion must be suggenic, and this occurs when the principles are not common to multiple genera, but proper; so it is clear that, in describing universality, the "nearness" of the middle term to the principle will be a nearness to the proper principles of the genus in question, not to some common principle. The point is that these universal principles are precisely not universal in the sense of being common to all sciences or having for its domain of discourse *all beings* or being "generally accepted";

[48]Aristotle, *Posterior Analytics*, 76a8-10. For other passages claiming that the conclusion and the (indemonstrable) first principles must be suggenic, in order to have a genuine demonstration, see, e.g., 75b11-13; 84b16-19; 87a40-87b4.

[49]Ibid., 88b27-28.

[50]Aristotle, *Aristotle's Posterior Analytics*, 86a15; a16.

rather, this universality is clearly indexed, or relative, to the underlying genus with which a demonstration is concerned.

On the other hand, Aristotle claims, "the more a demonstration becomes particular, the more it tends to sink into an infinity of things, whereas a universal demonstration tends towards the simple and towards that which has a limit."[51] This already suffices to make it clear that he does not regard the proper principles of a science, which act as indivisible limits, as "particular," but rather as universal. The degree of universality is said to be measured by the "nearness" of the middle term of the demonstration to the extreme term of an immediate proper principle; in the limit, then, the "most" universal in each genus would be given by those middles that are commensurate or coextensive with the extreme terms of the immediate proper principles of the relevant science. Since these immediate proper principles are regarded consistently as *indivisible limits* and as that which ensures the finiteness of demonstration, it makes sense that Aristotle would say that the more universal a demonstration is, the more it tends towards "that which has a limit," while the more particular it is, the more it tends towards "an infinity of things." In terms of our previous discussion of continuous demonstrations—built from the minimal collection of indivisible or immediate propositions sufficient for reaching the conclusion suggenic with the rest of the terms—it is clear that the most universal demonstration would be a continuous demonstration, but that this would always be indexed to a given subject genus.[52]

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[51]Aristotle, *Aristotle's Posterior Analytics*, 86a4-6.

[52]This initial sketch should also serve to indicate that Aristotle indeed had something like a "principled story to tell" about the different levels of generality and how this depends on the context.

When Aristotle speaks of principles as universal, then, he would appear to be thinking above all of the *proper principles* that pertain to a single science, for the common principles or axioms are shown in general to be insufficient to prove the conclusions regarding particular genera; and in those cases where they are sufficient, they prove "too much." Moreover, even if the principles of sciences could be reduced to some minimal collection, each science must have at least one principle for its genus, and conclusions in one genus are independent of the principles and conclusions demonstrated in another genus, which immediately implies (by *Suggenicity*) that the principles are independent of one another as well. In this way, the underlying genus of a science is not only the basis of the independence of the sciences, but the genus is even more fundamentally the basis of the determination of the scope of the proper principles of each science. A science is thus said to be *one*

> when it is of one genus of things, [and these are] all those [things] which are composed of the primary [elements] or are parts or are essential attributes of these. But one science is different from another if neither the principles of both comes from the same [elements] nor those of either [science] comes from those of the other science. We have a sign of this when we go back to the indemonstrables; for these must be in the same genus [ἐν τῶ αὐτῶ γένει] as that which is demonstrated. And a sign of this is the fact that the things demonstrated through them are in the same genus [ἐν τῶ αὐτῶ γένει] or are suggenic [συγγενῆ].[53]

We thus see that more fundamental than all the other assumptions—and as such pivotal to the resolution of the three main problems of the *Posterior Analytics*—is the assumption of *Suggenicity.* This importance is reinforced again in I.32, where Aristotle develops arguments against the notions that (1) "all syllogisms might have the same principles" and (2) that the principles might all come from the same genus yet "from one set of principles follows

[53]Aristotle, *Aristotle's Posterior Analytics*, 87a39-87b4.

one set of conclusions and from another set follows another set."[54] Again, appeal is made to the fact that the principles of things that differ in genus will be different in genus and cannot be applied to one another,[55] i.e., the *No Genus Crossing* requirement, which we know rests on the notion of *Suggenicity*. This suggenicity represents the unity of a given subject genus, a unity that, through its indivisible limits (its immediate proper principles), both "specializes" or "discretizes" the subject matter in question (rendering the science that treats it independent of other sciences), and also acts as the source of the necessity and internal coherence of a demonstration using the proper principles. The measure of the generality (or "relativized" universality) of a given premiss in that science is then given by the "nearness" of the middle terms to the immediate proper principles. Aristotle will also develop all this further in terms of the language of *causality*, claiming that "the universal is the first subject to which the attribute belongs and is therefore the cause."[56] In terms of how this is reflected in our own knowledge of a thing, he says that "we understand in the highest degree when the attribute belongs to a subject not in view of something further"[57] and to reach such a thing is to reach "an end and a limit."[58] "Most universal" in each genus will thus consist of the causes of per se relations that do not themselves have a further cause (because one would then have to extend beyond the limits of the genus itself). As for the necessity of a conclusion: this is carried by the middle or mediating link in a syllogism,

[54]Aristotle, *Aristotle's Posterior Analytics*, 88b24-25.

[55]See ibid., 88a33-34.

[56]Ibid., 85b27.

[57]Ibid., 85b37.

[58]Ibid., 85b31.

for we know that "demonstrative knowledge must be knowledge of a necessary nexus, and therefore must clearly be obtained through a necessary middle term."[59] In demonstrations, per se relations coincide with necessary relations, while accidental relations coincide with contingent relations. And we know that a predication will not be essential, and the relations will not be necessary, if the extreme and middle terms in a syllogism are not "drawn from the same genus."[60] Even the fundamental relation of *consequence* at play in Aristotle's account of deduction thus relies on his *Suggenicity* requirement. Recalling the "internal" continuity of demonstrations, this notion of suggenicity begins to suggest the close connection between continuity and a more flexible notion of generality.

We have observed that the proper principles of a science, revealing those indemonstrable facts of its subject genus, will consist of (i) the definitions of subjects and attributes, and (ii) existence claims regarding the subject genus. When Aristotle goes on to suggest a way of arriving at a definition of the essence or 'what it is' of something "by division," he claims the following:

> For constructing a definition through divisions, it is necessary to aim at three things: (1) grasping the things predicated in the essence, (2) arranging these in proper order, and (3) [being sure] that these are all.[61]

This procedure would appear to be nicely reflected in (our discussion of) the construction of continuous lines of predication. With regards to arranging the essential predicates in order, Aristotle has this to say:

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[59]Aristotle, *Posterior Analytics*, 75a12-13, translation modified.

[60]Ibid., 75b9.

[61]Ibid., 97a23-26.

> The proper arrangement will occur if one takes the 'first', and this will be the case if what is taken follows [i.e., is predicated of] all [the others], but all do not follow it—for there must be such a one. When this has been taken, the same method [is used] forthwith for the terms below. For that which is 'first' of the others will be second and what is ['first'] of the subsequent ones is third. For when the thing above is taken away, the next one will be 'first' of the others, and similarly for the rest.[62]

Thus, in "dividing," we must still aim to reach the 'first' propositions beyond which we cannot divide.[63] Deslauriers calls the special sort of definition at issue here—namely the revelation in a demonstration of an indemonstrable ground of what a thing is, revealing the uncaused cause—"immediate definition."[64] The immediacy of such definitions reflects the necessity of the connection between the object defined and the cause of that object, i.e., such objects are not distinct from their cause. It is the special sort of unity that belongs to the underlying genus being defined in terms of its first causes that gives these definitions the necessity that further enables them to act as principles for demonstration. But we saw that this unity is nothing other than the internal continuity of the genus, secured by the existence of certain immediate, indemonstrable proper principles, that further reflect the causal order in nature; those same principles, as indivisible limits, simultaneously mark off or "discretize" each genus (and the science that has this for its subject) as its own distinct type, beyond which boundaries a demonstration wanders at the price of having to content itself with only accidental relations.

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[62] Aristotle, *Posterior Analytics*, 97a28-34.

[63] In this use of 'first' in this connection, the reader should recall Definition 1.0.4.

[64] See Deslauriers, *Aristotle on Definition*.

When a parallel between the *Posterior Analytics* and the *Physics* has been noticed, I mentioned that it is typically remarked that Aristotle's responses to the two structurally similar problems are strangely dissimilar: he appears to be an "atomist" when it comes to demonstration, while in response to Zeno he "saves" change by carefully dismantling the atomistic account. We have begun to see how it is a misleading oversimplification to say that Aristotle is an "atomist" as regards demonstration. As we have started to see, and as the epigraph at the beginning of this chapter already suggested, it would be more accurate to say that Aristotle is an "atomist" about distinct kinds in relation to one another, but a "non-atomist" about the internal structure of each kind. While it may be superficially correct to say that the conclusions reached in the *Posterior Analytics* differ from those reached directly in response to Zeno, the *Posterior Analytics*'s framing of the main problems and their attempted resolution is not just entirely consistent with, but is closely aligned with, his wider account of continuity as presented throughout the entire *Physics* (of which the response to Zeno forms just a small part). In order to further defend this claim, and to clarify the emerging connection between continuity and *genos*, we turn now to the *Physics*.

Chapter 2

Aristotle's Physics Revisited

Physics Introduction

We are told, as early as *Physics* I, that every motion is between contraries (ἐναντία) of some sort. As *Physics* I.5 stated,

> all things that come to be come to be out of, and all things that pass away pass into, their contraries or intermediates. And the intermediates [are from] the contraries. For example, the colors come out of white and black. And so all of the things that come to be by nature are contraries or things that come to be out of contraries.[1]

Book II stresses that many motions are incidental (κατὰ συμβεβηκὸς) to the things they move or in which they move. Motion, Book III claims, appears (δοκεῖ) to be one of the continuous things (τῶν συνεχῶν);[2] moreover, motions can reduced to four kinds: changes of thinghood (οὐσία), quality, quantity, or place. Aristotle begins Book V by setting aside incidental motions, and claims to treat motion "in its own right"; this too is also said to be characteristically from one contrary to the other. But, beginning in Book V, Aristotle deepens and complicates his consideration of this fact, in the course of which it emerges that there is in fact no contrariety with respect to thinghood—and so coming into being and perishing must be changes from *contradictories* (αντιφασις), not contraries, and as such must constitute discontinuous changes. Thus, it turns out that while such changes were first taken to be of four sorts, Aristotle comes to conclude that motion in the strictest sense is

[1] Aristotle, *Aristotle*, 188b21-26, my translation.

[2] Ibid., 200b14-15.

only threefold, for only changes in quality, quantity, and place are truly between contraries. And these are the motions that are importantly continuous.

Understanding how and to what extent, for the remaining three types of motion, the contraries serve as boundaries of an in-between range that admits of continuous variation—and how, as "extremities" and limits, these contraries act as the principles and "completions" for the intermediate motions happening between the extremities, thereby providing their unity, and even, in certain ways, the unity of that which undergoes the motion—is arguably one of the main objectives of the rest of the *Physics*. Of paramount importance, then, is that we establish the links between continuity and contrariety, on the one hand, and that we understand the role of boundaries or limits in relation to continua, on the other.

To begin, then, we note that nothing without parts is moved, as Aristotle is keen to emphasize throughout the *Physics*. Everything moved, and moved in its own right, is continuous—and so nothing continuous is without parts. But *how* exactly do continuous things have parts? For, as *Physics* VI will argue at length, nothing continuous is *composed of* indivisibles. As every reader of Aristotle knows, Aristotle ultimately claims that continua have parts in a very particular way: potentially. However, this simple formula masks a wealth of problems, and in fact remains poorly understood on its own, so at least for some time, I deliberately bracket consideration of this point. For Aristotle, as will be seen, it is precisely indivisibles that are made responsible for securing the continuity of continuous things and motions. So while it is true that infinite divisibility is, in one very particular and restricted sense, a necessary but not sufficient condition for continuity, the issue of continuity-discreteness cannot be cleanly mapped onto that of divisibility-indivisibility (even once we apply the actual-potential distinction).

While Aristotle is at pains, throughout more or less the entirety of Books III through VIII, to secure the continuity of motions in various contexts, along the way he develops a sophisticated theory concerning how continuous things, including motions, are characteristically at once not *composed of* indivisibles, yet are *made* continuous by their indivisible limits. At a macroscopic scale, even the continuity of the motion of the cosmos as a whole is ultimately guaranteed, as Book VIII aims to show, only by the characteristic action of the unmoved mover in its role as a *indivisible limit* of all motion. And at less grand scales, the characteristic features of this subtle but pivotal relation between continua and their indivisible limits appear again and again. But how do indivisibles guarantee the *continuity* of continua? And how is this not inconsistent with the necessary infinite divisibility of continua and their non-composability (from indivisibles)? The fundamental problem here, as we will see, extends its reach throughout Aristotle's corpus, and indeed, deeply permeates his approach to many seemingly disparate problems. However, we can initially, and perhaps most directly, approach the problem by jumping right into Aristotle's (crowd-favorite) first discussion of time from *Physics* IV.

The Continuity of Time

Time, Aristotle tells us, *is* not motion, but is necessarily *of* motion.[3] Being *of* motion, time is equally determined positionally, after the manner of the "most ruling" of the motions, change of place; moreover, since magnitude is continuous (219a12), motion must be continuous, for motion follows magnitude (219a12-14), and through motion there is time. Now, as "before

[3]Aristotle, *Aristotle*, 219a7-10.

and after belong first of all to place, and thereby to position," and since "there is a before and an after in magnitude, it is necessary that there also be a before and an after in motion, analogous to those in the magnitude," so also "there is a before and an after in time."[4] In fact, whenever there is a motion, "there is a before and an after in it";[5] however, Aristotle notes, it is evident that this structure of having a "before-and-after" does not already, on its own, encapsulate all that motion (and thus time) is. Rather, we seem to recognize time most precisely

> whenever we mark off a motion, marking it off [ὁρίσωμεν] by means of a before and an after [in a motion][...]. And we mark them by taking them to be other than one another [ἄλλο καὶ ἄλλο], with something else between [μεταξύ] them. For whenever we think the terms [τὰ ἄκρα] [i.e., the 'before' and 'after'][6] differ from the middle, and the soul says there are two nows, one before and one after, then also we hold this to be time. For time seems to be bounded [ὁριζόμενον] by the now—let this be laid down. So whenever we perceive the now as one, and neither as before and after in a motion, nor the same but belonging to something before and something after, no time seems to have happened because no motion has. But whenever there is a before and an after, then we say there is time, for this is time: a number of motion fitting along [κατὰ] the before-and-after. Therefore time is not motion except insofar as the motion has a number.[7]

For our purposes, there are three things worth pointing out here: (1) that the before and after are "other than one another" in such a way that there is something *in between* them; (2)

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[4]Aristotle, *Aristotle's Physics*, 219a16-17; a18-19; a20-21.

[5]Ibid., 219a22-23.

[6]It is misleading of translators to translate τὰ ἄκρα here as "extremities," as Sachs and Wicksteed and Cornford do. While in many instances, Aristotle does use τὰ ἄκρα and τὰ ἔσχατα interchangeably, given the ultimate identification of the now with the extremities (τὰ ἔσχατα) of both the before and the after, it does not make sense to translate τὰ ἄκρα here as "extremities." Moreover, the context makes it clear that he simply means to refer to the two differing "terms" or "sections" of the time.

[7]Aristotle, *Aristotle's Physics*, 219a24-219b4.

that we have 'time' whenever "the terms [ἄκρα] differ from the middle"; and (3) that 'time' is *bounded* by 'the now'. On this account, 'the now' importantly plays a dual role, "existing in one way as always the same, but in another way as not the same," a role Aristotle elaborates on in his discussion showing that time is continuous "for it is of something continuous [συνεχοῦς γάρ]":[8]

> Time, then, is continuous by means of the now [συνεχής τε δὴ ὁ χρόνος τῷ νῦν], and is divided by the now [διήρηται κατὰ τὸ νῦν]. This also follows the change of place and the thing carried along [τῷ φερομένῳ]. For also the motion and the change of place are one in the thing carried along, because it is one (not just by being at one time, since it might make stops [διαλίποι], but also in meaning [τῷ λόγῳ]). And the now marks off [ὁρίζει] the motion into a before and an after; and this it does in a manner corresponding to that of the point. For the point also both holds together [συνέχει] the length and marks it off [ὁρίζει], since it is a beginning [ἀρχὴ] of the one [part] and an end [τελευτή] of the other.
>
> But whenever one takes it in this way, using what is one as two, it is necessary to make a stop [ἀνάγκη ἵστασθαι], if the same point is to be a beginning and an end. But the now is always other through the moving of the thing carried along. So time is a number not as of the same point, which is a beginning and an end, but rather as the extremities [τὰ ἔσχατα] of a line [form a number]; and neither is time a number as parts [ὡς τὰ μέρη], both because of what has been said (for one might use the intermediate point as two, so that time would happen to stand still), and further because it is clear that the now is no part of time, nor is the division part of the motion, just as neither is the point any part of the line, but it is two lines that are parts of one line. Then insofar as the now is a limit [πέρας], it is not time but an attribute [συμβέβηκεν] of time; but insofar as it numbers, it is a number. For the limits belong only to that of which they are the limits [τὰ μὲν γὰρ πέρατα ἐκείνου μόνον ἐστὶν οὗ ἐστιν πέρατα], but the ten which is the number of these horses belongs also elsewhere.[9]

[8]Aristotle, *Aristotle's Physics*, 219b33;220a26.

[9]Ibid., 220a14-24.

While much attention has been paid to Aristotle's definition of time as a number of motion, it is in fact the other sense—the now as a *limit* of time—that largely drives the discussion forward. In these passages, we begin to see that the now not only "*divides* the time into a before and an after," but also that it is "what is both a beginning of one part and an end of another," and as such "*holds together* [συνέχει] the length." In this latter way, *the now makes time continuous* ("time is continuous by means of the now"). Aristotle emphasizes the now's role of "holding together" in what follows:

> the now is the continuity of time [ἐστιν συνέχεια χρόνου], as was said; for it holds together [συνέχει] past time and the future. And it is the limit/boundary [πέρας] of time; for it is the beginning [ἀρχή] of one part and the end [τελευτή] of another.[10]

However, regarding its role as *dividing*, we already heard that "the now is no *part* of time, nor is the division *part* of the motion." Elaborating on this dual role of the now as both dividing and uniting (holding together), Aristotle notes that the now "divides potentially [διαιρεῖ δὲ δυνάμει], and insofar as it is a division, the now is always different, but insofar as it binds together, it is always the same, as with mathematical lines."[11] In this respect, Aristotle compares the now to a point: for a point is not always understood in the same way, since "as dividing, it is other and other, but insofar as it is one, it is the same in every respect."[12] Thus, the now plays a dual role: "the now is in one way a division of time, potentially [διαίρεσις κατὰ δύναμιν], and in another a limit of both and unity of its parts [τὸ

[10]Aristotle, *Aristotle's Physics*, 222a10-12.

[11]Ibid., 222a13-15.

[12]Ibid., 222a15-16.

δὲ πέρας ἀμφοῖν καὶ ἑνότης]."[13] And it is true that the dividing and the uniting are "the same act of the same thing [ἔστι δὲ ταὐτὸ καὶ κατὰ ταὐτὸ ἡ διαίρεσις καὶ ἡ ἕνωσις]," yet "the being [τὸ δ᾽ εἶναι] of them is not the same."[14] In these passages, we see a very clear statement of the now's dual role: the now is not just a *dividing*, but as the limit, the now is equally made to supply the *unity* of the parts into which it has divided the motion. But what exactly is the relation between the (unifying) now-as-limit and the (divided) time-as-limited? We know, for instance, that "the limits belong only to that of which they are the limits." But we have also seen Aristotle point out that "the now is no part of time, nor is the division part of the motion, just as neither is the point any part of the line, but it is two lines that are parts of one line."[15] Moreover, we know that times are always oriented by their *position* with respect to their "nearness to the indivisible now":[16] this positional nature accounts for how it is that the language of 'before' and 'after' is applied in relation to (and in separation from) the now, which acts as the boundary of the past and future determining, e.g., which 'befores' are before other 'befores', and how "in the past we call what is farther from the now before and what is nearer after,"[17] etc. But Aristotle also claims, importantly, that "that in which the now is, the separation from the now is also."[18] To begin to make sense of all this, let us first collect what we already know:

[13]Aristotle, *Aristotle's Physics*, 222a17-19.

[14]Ibid., 222a19-21.

[15]Ibid., 220a19-21.

[16]Ibid., 222b9.

[17]Ibid., 222b10-11.

[18]Ibid., 223a8-9.

1. There is time whenever the "sections/terms" differ from the middle, i.e., time is whenever *there is an in-between.*
2. The now divides a time into a 'before' and an 'after' (an ordering it ultimately derives from the positional nature of place); and the sections of 'befores' and 'afters' are ultimately ordered in terms of their separation from (or "nearness to") their "limit" (the now).
3. The now is the beginning (ἀρχή) of one part (say, the right section representing all of the 'after' stretches of time) and the end or completion (τελευτή) of the other part (say, the left section representing the 'befores')—as such, it is the common *limit* of each of the time stretches, and so is the unique limit of the before and after.
4. Via (2) and (3), the now has a dual function of dividing and uniting, but precisely in its manner of *uniting* or *holding together* what has been divided, in its role as limit, the stretch of time as a whole is made continuous.
5. But the now is indivisible.
6. And the now is no *part* of the time.

Thus, the now as limit is (i) indivisible; (ii) no *part* of that of which it is the limit; and (iii) precisely that by which the whole time is made continuous. At least at first, it seems that we should be somewhat troubled by all this. After all, we know that Aristotle will also argue, on many occasions, that time, *as continuous*, must always be divisible (something already anticipated by the first point's notion that *time is whenever there is an in-between*). Since what is here made responsible for the very continuity of any given time is itself explicitly singled out as indivisible, unless we find it perfectly acceptable to make something indivisible responsible for (at the very least) the infinite divisibility of something, it is clear that continuity cannot be identified with infinite divisibility—or, in any event, if continuity indeed (in one sense) requires but is not identical with infinite divisibility, clearly something more subtle is going on here.

Bracketing the issue of potentiality[19] allows us to focus more directly on the fact that whatever the indivisible limits are, they cannot belong to the thing they limit, if by 'belong to' we simply understand 'are a part of' (nor does it seem plausible that something indivisible would belong to anything "potentially"). On the other hand, given that Aristotle *does* on occasion say things to the effect that "the limit is in what is limited"[20] and "the limits belong only to that of which they are the limits,"[21] we are left somewhat at a loss as to what to say about the specific manner in which a limit carries out its action and how it is related to that which it limits. We can surely say that the indivisible now, as the limit of time, neither is *in* the time in the manner of a *part*, nor *composes* the time (anticipating the arguments of Book VI). Yet it is nonetheless precisely what *makes the time continuous*, and as such unites the parts into a whole. At least in one sense—yet to be determined, but evidently related to the fact mentioned in the previous sentence—it does seem to "be

[19]I am deliberately bracketing the fact that the now *divides potentially*—notice that he does *not* say that in *uniting* or *holding together*, the now does so *potentially*—both because this fact does not feature much in this present discussion, and because I do not believe that the distinction helps resolve the present issue. Moreover, on account of the dual role of the indivisible now, specifically in its role as *uniting* or *holding together*, if one were to assert that the now was a part of the time—just *potentially* a part—this is not even correct, since the now does not appear to be a *part* of the time *in any way*. Nor does it seem plausible, on its own terms, that in its manner of uniting or holding together, the now is in any way "potential." At least in these passages, in precisely its unifying aspect, the indivisible now would appear to be more closely aligned with actuality than potentiality. Moreover, of course, *everywhere* throughout Aristotle's corpus, it is the indivisible that is said to act as, and to be, a *limit*.

[20]For instance, at Aristotle, *Aristotle's Physics*, 212b30.

[21]Ibid., 220a21-22.

in" that which it limits. To get another perspective on this issue, let us turn to Aristotle's discussion of place from Book IV.

On Place

In the *Categories*, Aristotle holds place to be "among the continuous things."[22] However, it is in *Physics* IV that the subtleties in the alleged continuity of place are more fully addressed. In his treatment of place in *Physics* IV, Aristotle gradually advances towards "what truly belongs to [place] in its own right,"[23] first offering the definition that place is "primarily that which surrounds that of which it is the place, and in no way belongs to the thing."[24] As such, the primary place is "neither less nor greater than the thing," is "left behind by each thing and is separate"; moreover, "all things having place have the up and the down, and each of the bodies is carried by nature to and remains in its proper place."[25] Aristotle's first attempt at a definition thus yields that place is the limit of the surrounding body (τὸ πέρας τοῦ περιέχοντος σώματος). He goes on to point out that

> whenever what surrounds is not divided but is continuous, a thing is said to be in it not as in a place but as a part in a whole; but whenever it is divided and touching, the thing is first of all in the innermost part of what surrounds it, which is neither part of the thing in it nor greater than its extension, but equal to it, for the extremities of things which touch coincide. If the surrounding thing is continuous, a thing is moved not in it but with it, but if the surrounding thing is divided, a thing is

[22]Aristotle, *Aristotle*, 5a8.

[23]Aristotle, *Aristotle's Physics*, 210b39.

[24]Ibid., 211a1-2.

[25]Ibid., 211a3-5; a6-7.

> moved in it, and no less so whether the surrounding thing is moved or not.[26]

Some examples of things that are not divided, but are rather held to be "as parts in wholes," are the eyeball in the eye and the hand in the body; an example of things that are divided are the wine in the jug—for, in the first case, the hand is moved with the body, yet in the latter case, the wine is moved in the jug.[27]

Again, we glimpse the subtle but important dual role of the limit as both unifier (of a whole) and that which marks a division. The main result of this section comes when Aristotle shows that while place must be either (1) the form, (2) the material, (3) "some sort of extension between the extremities," or (4) "the extremities if there is no extension besides the magnitude of the body present within,"[28] it cannot be the first three. And so, Aristotle concludes, place must be "the boundary of the surrounding body [at which it conjoins with the surrounded one] [τὸ πέρας τοῦ περιέχοντος σώματος [καθ ὃ συνάπτει τῷ περιεχομένῳ]],"[29]

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[26]Aristotle, *Aristotle's Physics*, 211a31-211b2.

[27]Ibid., 211b3-7.

[28]Ibid., 211b8-9.

[29]The "at which the limit conjoins with the surrounded body" (212a6-7) within the brackets may or may not have been added to the original text. Lang claims, following Ross, that these words appear only in the Arabo-Latin translation and that "Given their complete absence from the Greek tradition, they are probably spurious and should not be retained, as reflected by both Carteron and Prantl. Furthermore, not only are they unnecessary to the meaning of this text, they are not confirmed by any other passage in the account of place" (92). I agree with Lang's important point that while συναπτει is often translated here by "in contact with" or "touching," implying that place and what is surrounded are in fact two bodies, this cannot be right:

> There are two problems here. (1) For "in contact with" in the sense of one body in contact with another, Aristotle usually uses ἅπτεσθαι. συνάπτει is clearly stronger; I translate it as "conjoin", but it often seems to mean "unite directly" or "coincide exactly." (2) As a limit, place resembles form and cannot be another body. Indeed, Aristotle compares place and form as limits and contrasts place as a limit with body. A limit contrasts in several ways with what is limited: it is a formal constitutive part, indivisible, more closely identified with substance and more honorable. What

to which he adds that he means "by the surrounded body that which is movable with respect to place" (212a6-8). Building on this, he proposes his revised definition:

> place is meant to be motionless, on account of which it is rather the whole river [through which a boat moves] that is a place, since the whole is motionless [ἀκὶνητον]—therefore, this is place: the first motionless boundary of what surrounds/contains [τό τοῦ περιέχοντος πέρας ἀκίνητον πρῶτον],[30] and the place coincides with the underlying thing, for the boundaries coincide with the bounded [ἔτι ἅμα τῶ πράγματι ὁ τόπος· ἅμα γὰρ τῶ πεπερασμένω τὰ πέρατα].[31]

Beginning to address the issue of whether or not place is itself in a place, and whether all things are in place, or even the "whole" cosmos in place, Aristotle first proposes that some things are in a place potentially, while others are in a place actually. He clarifies this distinction by saying that "whenever something homogeneous is continuous [συνεχὲς ᾖ τὸ ὁμοιομερές], the parts are in places potentially [κατὰ δύναμιν ἐν τόπω τὰ μέρη]," but "whenever the parts are separated but touching [χωρισθῇ μὲν ἅπτηται], like a heap, they

> is limited, in contrast to a limit, is a body capable of motion. Hussey asks why Aristotle specifies body as movable here: because the specification of place as a containing limit and the contained as a movable body reveals the relation between them — and so the meaning of συνάπτει, conjoin. Place and movable body are not conjoined as two bodies in contact, but as a limit (place), and what is limited (movable body). However, Aristotle has already asserted that the conjunction between container and contained most nearly parallels that of form and matter: the limit and the limited are conjoined as constitutive principle and that which is constituted.[. . .] [T]he limit and the limited together comprise one being.(Lang, *The Order of Nature in Aristotle's Physics*, 92-93.)

[30] "First" here does not mean "nearest" (to the contained body), but, as in *De Caelo*, "[While] the middle is a source and precious, the middle of place seems last rather than a source; for the middle is what is bounded and the limit is the boundary. And the container, namely the limit, is more precious than the limited; for [the limited] is the matter while [the limit] is the thinghood [οὐσία] of the system" (*De Caelo*, 293b11-15). Lang cites this passage as well, claiming that "The first motionless limit is the substance of the cosmos because it renders the cosmos determinate rather than indeterminate. By rendering the cosmos determinate, the limit makes it a whole and defines every part within the whole as "up", "down", and "middle"" (ibid., 101).

[31] Aristotle, *Aristotle's Physics*, 212a20-31.

are actually so [κατ ἐνέργειαν]."[32] In general, whatever is somewhere, i.e., in place, both "is something itself and needs there to be something else besides it, in which it is and which surrounds it."[33] However, aside from the cosmos as a whole or the sum of all things, nothing can be outside the sum of all, for which reason "all things are in the heavens, for the heaven is equally the whole—yet it is not the case that heaven is place; rather some boundary of it is the place of the movable body it is touching."[34]

This account of place involves many difficulties—difficulties that, for our purposes, we need not resolve. As far as Aristotle is concerned, he takes this account to resolve the *aporiae* of place from which he began (212b25), and the passage in which he sums this up will be of some significance for where we are headed (in addition to looking back to the previous chapter's discussion of the *suggenic*):

> For [on this account] the place need not grow along with [συναύξεσθαι] a thing, nor a point have a place, nor two bodies be in the same place, nor need there be some sort of bodily extension [διάστημά σωματικόν] (for what is within the place is the body that happens to be there, but not an extension of body). *And place too is somewhere* [πού], *but not as in a place, but rather as the limit* [πέρας] *is in what is limited.* For not every being is in a place, but only movable bodies. And it is reasonable that each thing is carried to its own place [φέρετα δὴ εἰς τὸν αὑτοῦ τόπον ἕκαστον]. For what succeeds something and is touching it [ἐφεξῆς καὶ ἁπτόμενον], not by constraint [μὴ βία], is of the same genus [συγγενές], and things fused together are not acted on by one another [συμπεφυκότα μὲν ἀπαθῆ], while those that touch [ἁπτόμενα] are able to act on, and be acted on by, one another [παθητικὰ καὶ ποιητικὰ ἀλλήλων].[35]

[32] Aristotle, *Aristotle's Physics*, 212b3-8.

[33] Ibid., 212b10-13.

[34] Ibid., 221b17-22.

[35] Ibid., 221b27-38, my emphasis.

We will have occasion to return to some of the aspects of this account of place in what follows, and passages such as the above will be shown to be part of a larger strategy. For now, though, I will compare some of the general features of this account to that given of time and the now.

In the case of place, the contained/surrounded is not contained by another *body*, but the contained is in the limit *in the way the limited is in the limit.* It is important to realize that the limit (as place) is not in the same *place* as the contained, but rather *constitutes* the place without itself being in place, just as it contains a body in motion without itself being a body in motion. As a limit, place does not have the characteristics of what is limited—otherwise, it would be a *part* of that which it limits. Moreover, place, as limit, is itself unmoved and, as such, cannot be identified with matter. Paralleling this account of place, just as place is somewhere, not as "in a place" but rather as "the limit is in the limited," the now is also "some-when" and is the first motionless boundary (of the before and after), but the now does not have a time of its own (in fact, as we will see, Aristotle will explicitly argue later on that it does not). The now is "some-when," not as *in* a time, but rather *as the limit is in what is limited.* This distinction appears in multiple contexts and is clearly of great importance. However, we seem to be back to where we started. For, what does it mean for a limit to be *in* what is limited?

In *Physics* VI, Aristotle returns to the special question of the indivisibility of the now, and its relation to the time it limits and renders continuous; but the resolution of this issue comes only after, and on the basis of, the elaborate account of continuity given in Books V and VI. It is by means of the carefully crafted concept of continuity developed there that he aims to resolve these specialized concerns and others besides, and he does so by primarily addressing the most general form of the question we have thus far raised in the restricted contexts of time and place: how all changes and beings that support change can be continuous and are *made* continuous, or are *constituted* as continuous, by virtue of their indivisible limits, without thereby being *composed of* indivisibles.

Continuity Defined

Physics V

Having considered the "common things" that "follow upon" motion—more specifically, what follows from the continuity of motion, such as place, time, and infinity[36]—Aristotle commences a treatment of motion taken at a still more general level. Book V begins by claiming that change occurs in three ways: (1) accidentally or incidentally; (2) in a part; (3) in its own right. Aristotle is often eager, throughout these books, to leave aside accidental change and partial changes, and to focus on an articulation of motion in its own right. Such non-accidental and non-partial motions are said to be "not in everything but rather in contraries and in what is between them," something we are led to believe on account of the many examples where "a thing changes from what is in-between, since it is used as being contrary to either extreme, for the in-between in a certain way is the extremes."[37] However, while in previous books Aristotle spoke of changes of thinghood, quality, quantity, and place, this demand that motion must always be from one contrary to the other means not only that "the

[36] I am referring to Aristotle's claim, at the beginning of Physics III (200b16-27), that he plans to address not only motion but "the things that follow upon motion," or more specifically, since "motion seems to be one of the continuous things," those things that follow from the continuity of motion, like infinity.

[37] Aristotle, *Aristotle's Physics*, 224b31-32; b33-35. The claim that "the in-between in a certain way is the extremes" will be addressed below in more detail in the "Interlude" section.

change from one subject [ὑποκειμένου] to another is the only motion [in its own right]; and the subjects are either contraries or in-between (for let the deprivation [στέρησις] be set down as a contrary),"[38] but also that there are in fact only three such motions: that of quality (τοῦ ποιοῦ), that of quantity (τοῦ ποσοῦ), and that with respect to place (κατὰ τόπον).[39] As for motion with respect to thinghood (οὐσία): "since there is not among beings a contrary to an independent thing,"[40] in truth we cannot call this a motion; coming-into-being thus represents a discontinuous change from one contradictory to the other. Thus, while change remains of four types, motion in its own right is strictly confined to changes with respect to quality, quantity, and place, for precisely the reason that only these are changes "between contraries," only the latter are continuous, and "motion is one of the continuous things." The extremities of a motion, then, must be boundaries of an in-between range of conditions, something that contradictories (via the excluded middle) cannot support; the three sorts of motions listed above are thus the only changes that admit of continuous variation in this way, and so they are the only things that deserve to be called motions.

The initial description of motion with respect to quality—or alteration (ἀλλοίωσίς)—that immediately follows will be of some interest to us in subsequent sections and will occupy us to a much greater extent in Chapter 3, so I take the opportunity to record it now. Motion with respect to quality pertains to that by which something "is said to be affected or unaffected [πάσχειν ἢ ἀπαθὲς εἶναι]," and thus is

[38] Aristotle, *Aristotle's Physics*, 225b3-5.

[39] Ibid., 225b9-10.

[40] Ibid., 225b10-11.

> change within the same form [ἐν τῶ αὐτῶ εἴδει] to more or less [ἐπὶ τὸ μᾶλλον καὶ ἧττον], for it is motion either from a contrary or to a contrary, either simply or in some particular way [ἢ ἁπλῶς ἢ πή]. For change to a lesser degree of a quality will be said to change to the contrary, but what changes to a greater degree will be said to be changing from the contrary of that quality to the quality itself.[...] [A]nd the more or less is the greater or lesser presence in it [ἐνυπάρχειν] or absence from it of the contrary.[41]

Motion with respect to quantity, for its part, is called increase or decrease—more specifically: "motion in the direction of completed magnitude is increase [εἰς τὸ τέλειον μέγεθος αὔξησις], while motion in the contrary direction is decrease."[42] Finally, motion with respect to place is "without name," but can be called locomotion (φορὰ) in common.[43]

Immediately following this clarification of the sorts of motion that deserve to be called motion in its own right, Aristotle commences (in V.3) his extended discussion of continuity. Since all motions "seem to be continuous," and since he has been considering those types of motion that involve motion in its own right, it is entirely natural that he should provide a treatment of continuity on its own, for the continuous is "that of which the motion is one in its own right." Let us begin by presenting Aristotle's rather dense definition from Book V without any preparation:

> The continuous is that which is contiguous, but I call things continuous only when the limits [πέρας] at which they are touching become one and the same [ταὐτο γένηται καὶ ἓν], and, as the name implies, hold together [συνέχηται]. But this is impossible if the extremities [ἐσχάτοιν] are two. This definition makes it clear that the continuous belongs to those things out of which some unity [ἐξ ὧν ἕν τι] naturally arises

[41] Aristotle, *Aristotle's Physics*, 226b2-8.

[42] Ibid., 226a31-35.

[43] Ibid., 226a30.

> [πὲφυκε γίνεσθαι] in accord with their conjoining [κατὰ τὴν σύναψιν]. And in whatever way [ὥς ποτε] the continuous becomes one [γίγνεται τὸ συνέχον ἕν], so too will the whole [τὸ ὅλον] be one, such as by a rivet [γόμφῳ] or by glue [κόλλῃ] or by ligament [ἁφῇ] or by growing into one another [προσφύσει].[44]

This definition involves a large number of highly technical terms, many of which Aristotle carefully defines throughout this chapter, as well as a number of dependencies among those terms, relations Aristotle is equally concerned with clarifying. The definition just given emerges, in fact, as the last of three closely related notions—the other two being 'the successive' and 'the contiguous', each of which had in turn been constructed on the basis of the still more primitive terms: 'the coincident', 'the separate', 'the in-between', 'the touching'. Let us thus begin to unpack the definition by providing definitions of the latter four notions:

Definition 2.0.1. Coincident/together (τὸ ἅμα): "things are said to be coincident in place when they are in one primary/first place [ὅσα ἐν ἑνὶ τόπῳ ἐστὶ πρώτῳ]."[45]

Definition 2.0.2. separate (χωρίς): "whatever things are in different [places] [ἐν ἑτέρῳ]."[46]

Definition 2.0.3. touching (τὶ τὸ ἅπτεσθαι): "those things of which the extremities are coincident [ἅπτεσθαι δὲ ὧν τὰ ἄκρα ἅμα]."[47]

Definition 2.0.4. in-between (τί τὸ μεταξὺ): "that at which a changing thing, if it changes continuously [συνεχῶς], naturally arrives before it changes [εἰς ὃ πέφυκε πρότερον ἀφικνεῖσ-

[44] Aristotle, *Aristotle's Physics*, 227a10-18.

[45] Ibid., 226b23-24.

[46] Ibid., 226b24.

[47] Ibid., 226b25.

ϑαι] to what is by nature last [or: the extremity] [εἰς ὃ ἔσχατον μεταβάλλει κατὰ φύσιν]."[48] Aristotle elaborates on this last definition of the 'in-between', a definition that turns out to be extremely rich and pivotal, in a passage worth citing in full:

> Thus, the in-between involves at least three things, since in change it is the contrary that is the last [or: extremity] [ἔσχατον]. And since every change involves opposites, while opposites are either contraries or contradictories, but with contradictories there is no middle, it is clear that the in-between will be among contraries. And something is moved continuously [συνεχῶς κινεῖται] if it leaves no gap [διαλεῖπον] at all, or the least possible [ὀλίγιστον], in the underlying reality [τοῦ πράγματος][49]—not in the time (for a gap in time does not prevent things having a between, while, on the other hand, there is nothing to prevent the highest note's sounding immediately after the lowest) but in the underlying reality [πράγματος] in which it is moving. This is true not only for locomotion but for every other kind of change as well. What is contrary with respect to place is what is most distant in a straight line: for the shortest distance is finite/limited, and that which is finite/limited constitutes a measure.[50]

We will have occasion to return to many aspects of the definition of 'in-between' throughout the remainder of this chapter. For now, a few things should be emphasized: (1) the 'in-between' is that at which a changing thing—*if it changes continuously*—"naturally" arrives before having changed to what is "by nature" last (or the extremity); (2) in such changes, it is the *contrary* that acts as the last (or extremity); and since the contraries between which a change occurs are always at least two, (3) the 'in-between' will involve at least three things.

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[48] Aristotle, *Aristotle's Physics*, 226b26-28.

[49] While the πραγμα here is usually translated as 'thing', and it does indeed have something of the same freedom or wideness of scope that belongs to 'thing', here it appears to be used to refer to the underlying genus (in the sense developed below, not in the narrow sense) with respect to which the change occurs. I translate it as 'underlying reality' so as not to prejudice one into thinking that Aristotle is speaking of independent substances here, which he does not appear to be doing. If nothing else, the fact that thinghood is not even capable of motion in its own right, should suffice to convince one that Aristotle is not thinking of things, in the sense of independent substances, at this moment.

[50] Aristotle, *Aristotle's Physics*, 226b29-39.

Explaining the condition *if it changes continuously*, Aristotle remarks that (4) something is moved continuously when it leaves no gap, or the least possible one, in the underlying thing in which the motion occurs—*not* in the time, the continuity or discontinuity of which is not sufficient to determine the continuity of the underlying reality one way or the other. And (4) holds, he adds, for *every* kind of change.[51]

Building on the four definitions given above, Aristotle offers a definition of what is 'in succession', followed by what is 'contiguous':

Definition 2.0.5. in succession (τὸ ἐφεξῆς) is that which

> [1] being after the beginning [μετὰ τὴν ἀρχὴν] in position [ϑέσει] or in form or in any other respect that is similarly determinate [ἄλλω τινὶ οὕτως ἀφορισϑέντος], [2] has nothing of the same genus [τῶν ἐν ταὐτῶ γένει] between it and that to which it is in succession (by which I mean a line or lines if it is a line, or if it is a unit, a unit or units, or if it is a house, a house, but nothing prevents something else [i.e., of a different genus] being between). For [3] that which is in succession is in succession to something [ἐφεξῆς τινὶ ἐφεξῆς], and is posterior to it [ὕστερόν τι]; for one is not in succession to two, nor the first day of the month to the second, but in each case the latter is in succession to the former.[52]

Let me highlight the three main aspects of this definition: (1) what is in succession is that which follows after the ἀρχὴν—either in position, form, or "in any other respect that is similarly determinate." It should be emphasized that Aristotle explicitly *denies* that only what can be in *position* can be successive, requiring instead that the successive simply be

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[51]That "it leaves no gap" means that for such πραγμα, change does not skip over anything "in between" in passing from one contrary to the other. By contrast, whatever changes are bounded by contradictories (i.e., coming to be and passing away) will have no underlying πραγμα capable of supporting such intermediate states.

[52]Aristotle, *Aristotle's Physics*, 227a1-9.

"after" an ἀρχὴν in *any determinate respect.* (2) While nothing prevents something that is *not of the same genus* being in between successive things, it is characteristic of successive things that there not be anything *of the same genus* between it and that which it succeeds. By virtue of this condition, in an important (but restricted) sense, namely with respect to a single genus, *whatever is in succession will not have in betweens.*[53] (3) Succession is antisymmetric, a feature imposed by the order of priority-posteriority. It is also worth noting that in this definition Aristotle must mean "*immediately* succeeds," and not just "succeeds," for the latter would also hold for things that were ordered in terms of prior-posterior but for which there could obviously be intermediates of the same genus (the very thing he denies of them in (2)). The next definition builds on this one.

Definition 2.0.6. Contiguous (τὸ ἐχόμενον) is "that which, being in succession to something, is touching it."[54]

In other words, the contiguous is whatever is *both* (1) in succession (see 2.0.5) and (2) touching, i.e., has extremities that are together (see 2.0.3). Before coming to the definition of the continuous, it is important to appreciate that the definition of contiguous builds on that of the successive, just as that of the continuous will build on that of the contiguous. What is in succession was the first to appear in this account,

> for what is touching [τὸ ἁπτόμενον] is necessarily in succession [ἐφεξῆς], but not everything that is in succession is touching (thus succession is

[53]The alert reader will already realize that given that whatever is continuous is automatically in succession (as we shall see), there will be an important sense in which whatever is continuous does not have in-betweens. Such a reader might then readily appreciate that the discussion of continuity in *Posterior Analytics* was perfectly in line with the characterization to which he is building in the *Physics*.

[54]Aristotle, *Aristotle's Physics*, 227a9-10.

> found among things prior in definition, like numbers, while touching is not). And if things are continuous, they are necessarily touching, but if touching, not necessarily continuous; for their extremities [τὰ ἄκρα] may very well be together [ἅμα] without necessarily being one [ἓν εἶναι αὐτῶν], but they cannot be one without necessarily being together. So being fused together [σύμφυσις] is last to come about, for the extremities must necessarily have been touching if they are to be fused together [συμφύσεται]; but not all things that are touching have fused together [συμπέφυκεν], though among things that are not in contact [ἁφή], there is obviously no fusion of them together [σύμφυσις] either.[55]

So, if touching, then necessarily in succession; but not necessarily touching, if successive. If continuous, then necessarily touching; but not necessarily continuous, if touching. It is of course immediate that if continuous, then necessarily successive. This dependence was already explicit in the very first part of the definition of the continuous with which we began, namely as "something of the contiguous."[56] Building on the definitions of 'successive' and 'contiguous' (which themselves made use of the four prior definitions), we can now return to the definition of continuity, expanding it to more explicitly read:

Definition 2.0.7. Continuous is that which not only

- (1a) has extremities that are together, i.e., in one primary place (read: touch), and

[55] Aristotle, *Aristotle's Physics*, 227a19-28.

[56] Many otherwise careful commentators seem to overlook or forget such passages and the important order it establishes between concepts. This leads to erroneous claims such as, e.g., Lang's claim that "If a thing is continuous, its parts are in place potentially, because *in a continuous substance the conditions of being in succession and touching are not met*. But something that is continuous may be divided with the parts touching; when they are divided and touching, they are in place actually (212b4-6)" (Lang, *The Order of Nature in Aristotle's Physics*, 115, my emphasis).

- (1b) has nothing of the same genus between what succeeds and that which it immediately succeeds (in relation to which it is ordered as posterior-prior) (read: is in succession),

but also that for which (2) the limits (πέρας) at which the extremities (ἐσχάτοιν) touch

- (2a) become one and the same (ταὐτο γένηται καὶ ἓν)(i.e., there is *a* limit, not limit*s*) and
- (2b) hold together (συνέχηται).

We recall also, from the second half of the original definition, that we were given some additional conditions and corollaries, which are worth recording separately:

Corollary 2.0.7.1. The continuous is that out of which some one/unity (ἐξ ὧν ἕν τι) naturally arises (πὲφυκε γίνεσθαι) in accord with conjoining (κατὰ τὴν σύναψιν). (*The Naturality Condition*)

At least in the *Physics*, this naturality is typically opposed to whatever arises "by force" (βία). In other contexts, though, Aristotle opposes it to whatever arises "by craft" (τέχνη).[57]

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[57]Compare this to *Metaphysics* VII.16: "Still, whenever the parts are one and continuous by nature, and not by force or by growing parasitically, they will all have being as potencies; for the exceptions are defective instances" (1040b13-15). Consider also that the "natural" unity achieved by the continuous is not, for Aristotle, either "automatic" or some kind of total regularity or stasis of parts melded together in such a way that all their distinctness disappears, as the following passage makes very clear:

> Also, what obstructs something from moving or acting by its own impulse is said to have hold of it, as columns have hold of the heavy things pressing down on them, and as poets make Atlas hold heaven, as though it would fall down to earth, just as some of the writers on nature also say. And in this way too what is continuous (τὸ συνέχον) is said to hold together what it connects (συνέχει ἔχειν), *as though it would be separated apart by each part's own impulse* (ὡς διαχωρισθέντα ἂν κατὰ τὴν αὑτοῦ ὁρμὴν ἕκαστον). (Aristotle, *Metaphysics*, 1023a22-25.)

Corollary 2.0.7.2. In whatever way (ὥς ποτε) the continuous becomes one (γίγνεται τὸ συνέχον ἕν), so too will the whole (τὸ ὅλον) be one. (*Oneness of Continuity Generates Oneness of Wholeness*)[58]

At least initially, the definition of continuity might seem to do more to draw attention to certain problems than it does to provide anything like a starting-point for their solution. To begin with: given that the limits at which the extremities touch (per (1a)) become one and the same (per 2a) and hold together (per 2b), what exactly do (2a) and (2b) add that (1a) (touching extremities) plus (1b) (that nothing of the same genus can be in between what is in succession) does not already provide? Moreover, how can the continuous even be a subclass of the contiguous? It would appear, at least at first, that contiguous things require the "twoness" of their extremities, even while they are touching, whereas continuity,

[58]Note also that Aristotle is clear at many points (see, e.g., Meta V.26) that while "wholeness is a certain kind of oneness" (1024a1), as long as something is continuous, even if it is not strictly *one*, it can be a whole. Similarly, completeness "in its own right," as that which has "nothing outside it" (1022b32), is possible only if we already have that the thing is continuous. In the contexts where Aristotle claims these dependencies (on which more below), he is not always clear as to why this is the case. However, we will see that it is because it first of all makes sense for there to be "nothing outside of" something that is continuous in its own right—for anything that was "outside of" the entity in question would have to be "outside of" its genus, and we know that "there is no way from one genus to another." It is precisely the work of the limit to accomplish this closure or completion. In connection with the above corollary, it is worth comparing this with the definition of "whole" from *Metaphysics* V.26:

> A whole means that of which no part is absent out of those of which it is said to be a whole by nature, or that which includes what it includes in such a way that they are some one thing. The latter can happen in two ways: either such that each is one or so that the one thing is made out of them. For a universal, which is attributed to a whole class of things as though it were a certain whole, is universal in the sense that it includes many things by being predicated of each, and that they are all one each-by-each, as are a human being, a horse, and a god, because they are all living things. But *what is continuous and finite* [!] is a whole whenever some one thing is made of a number of things, most of all when they are distinct constituents of it only potentially, *but if not, actively as well.*[...] Also, what has quantity has a beginning, middle, and end, and any quantity in which position makes no difference is spoken of as 'all', while any in which it does make a difference is called a whole [...].

of course, would demand that they be one. Put otherwise: continuity requires not only that the extremities "touch," but it more specifically demands the "fusing together" of the limits at which the extremities touch into one limit; yet it is not clear, on the face of it, how something that is 'one' can continue to be said to 'touch' itself. Perhaps even more troubling, since continuous things are necessarily successive, is that we are forced to try to make sense of how continuous things can indeed remain successive (in addition to whatever else they might be over and above successiveness), given that things in succession admit no in-betweens. On the face of it, this might seem difficult to countenance: for, successive things are defined as admitting nothing of the same genus in between, whereas even the very definition of the 'in-between' stipulated that in-betweens belong characteristically to what is continuous and to all sorts of changes that are not changes from one contradictory state to another. In this connection, it is initially unclear why the 'in between' was itself defined as it was, namely as belonging specifically to *continuous changes*. At the moment, it remains unclear how the continuous must at once characteristically have nothing of the same genus between those of its parts ordered by succession and be the very thing that will support 'in-betweens'.

Once these problems have been fully appreciated, they can be seen to go to the very core of Aristotle's *Physics*. To begin to approach the core of the difficulty, let us recall a passage already cited during our brief discussion of place:

> And it is reasonable that each thing is carried to its own place [φέρεται δὴ εἰς τὸν αὑτοῦ τόπον ἕκαστον]. For what succeeds something and is touching it [ἐφεξῆς καὶ ἁπτόμενον] [i.e., is contiguous], not by force [μὴ βίᾳ] [i.e., is additionally continuous, by the *Naturality Condition*], is suggenic [συγγενές], and things fused together are unaffected by one another [συμπεφυκότα μὲν ἀπαθῆ], while those that touch [ἁπτόμενα] are

> able to act on and be acted on by one another [παθητικὰ καὶ ποιητικὰ ἀλλήλων].[59]

Most notable here is the claim that touching things are able to affect one another, while continuous things—which, of course, as continuous, and so automatically contiguous, are already touching—would appear to be comprised of things that, belonging to the same genus and being thus involved in a kind of "fusion," become characteristically *unaffected* by one another. For now, though, I mean to draw attention to the fact that as long as the "succession plus touching" (contiguity) is not "by force," contiguous things become continuous precisely in being *suggenic* (συγγενές). We begin to see how the *Naturality Condition* (see 2.0.7.1)—as involving an active holding together of what it connects "as though it would be separated apart by each part's own impulse"—is bound up with "suggenicity," which invites us to consider that it is on account of this *suggenicity* of the candidates for continuity that the limits at which their extremities touch can become *one*, thus giving us continuity over and above contiguity, without thereby violating the stipulation that continuous things are also automatically successive and touching. In the section on the *Posterior Analytics*, we began to see how, initially, suggenicity was specified as a requirement that the "in between terms" and the extremities were of the same genus. By virtue of this requirement, the indivisible principles could be necessarily related to that which was demonstrated on their basis, for the requirement of suggenicity was nothing other than a stipulation that the limits (the indivisible, indemonstrable principles) and the limited (that which was demonstrated from the principles) in a demonstration belonged to the same genus, and only what belonged

[59] Aristotle, *Aristotle's Physics*, 212b27-38.

to the same genus could be shown to be related *per se* and necessarily. Moreover, in that context, we saw how the continuity of a proposition or a chain of propositions linked together into a demonstration was defined not only in terms of this suggenicity but in terms of the requirement that each immediately successive proposition shared a limit and so admitted nothing of the same genus in between, i.e., no further middle term that might amplify the demonstration.

This analysis could already suggest how to resolve our present problem. However, at this point, in the context of the *Physics*, we do not have a complete understanding of what it means for one part of a continuous thing, as suggenic, to be continuous (and thus also contiguous or successive and touching) with another, so there remains more work to be done. For now, note that not only is it the case that being "fused together" into a continuous thing is somehow something derived from, or at least closely connected with, the unity provided by "being in the same genus," but this suggenic characteristic of continua is what may even be responsible for the component parts of the continuum being characteristically "apathetic" to one another.[60] Leaving aside for the moment the exact implications of suggenicity, we

[60] I return briefly to this latter point later on in the chapter. Here, though, I can observe that this same point about "apathy" occurs in very different contexts. For instance, in *Metaphysics* (1046a28-29): "Thus, to the extent that something is fused together, it itself is not acted on by itself, because it is one thing and not different." In *De Anima* III.5, Aristotle claims that *nous* is *apathe*, since its *ousia* is *energeia* (he also calls nous *apathes* at 408b29); moreover, in the *Physics*, the first mover is also frequently spoken of as *apathe*. However, one should not be misled by this. For Aristotle also clearly holds, as in his discussion of *dunamis* in the *Metaphysics*, that

> all states [ἕξεις] in virtue of which things are altogether impassive [ἀπαθῆ ὅλως] to change or unchanging, or are not easily changed for the worse, are called potencies [δυνάμεις]. For things are broken and crushed and bent and in general destroyed, not because they have a potency, but because they do not have one and are deficient in some way. And things are ἀπαθῆ to such processes when they are hardly or slightly affected [πάσχει] by them because they have the potency [δύναμιν] and the ability [δύνασθαι] to be in some definite state [τῶ ἔχειν πώς]. (Aristotle, *Metaphysics*, 1019a27-30.)

can add to our main definition of continuity the fact that what is continuous will require suggenicity. Given the necessarily successive nature of the continuous, this will occur in such a way that not only is there nothing of the same genus in between the successive elements of the continuum (see 1b above), but (1c) as suggenic, the components of the continuous thing will (somehow) be so "fused together" as to become strictly unaffected by or "impassive to" one another.

Whatever else this definition does, it is should be clear that continuity in the sense developed in *Physics* V is primarily about what makes something—specifically something that is irreducibly a *plurality*—count as "one" or a "unity." It can only result in incoherence if one simply conflates this definition with the provisional notion of continuity as infinite divisibility from *Physics* III. Aristotle is clear that the *division* of something, whether accidentally or in itself, makes what may have been a unity into a 'many'. What the notion of continuity as infinite divisibility has to do with the present and more demanding sense of continuity as productive of a unity, is another matter (on which more below); but in the present context, note that in *Physics* VI and beyond, Aristotle will make it clear that *infinite divisibility is in no way a sufficient condition for making something continuous* (in the strong sense of Book V), though *in one sense* it does form a necessary condition. Closer inspection of the carefully constructed definition of *Physics* V gradually reveals that the initial uncertainties we have isolated in the definition stem, in large part, from not being adequately clear about the notion of the 'in between' on the one hand, and on the other hand from not sufficiently

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In this context we see that such "impassiveness" to change, or *invariance* (either completely or "not easily for the worse")—something that was attributed to continuous things—is precisely aligned with the having of a definite potency.

appreciating the dual nature of the limit of continua as that which not only divides but more importantly *completes what it limits and unifies parts that otherwise would separate by each part's own impulse.* Resolving these confusions will in turn yield a more careful and systematic account of how it can at once be true that (i) nothing of the same genus is in between the (immediately) successive components of continua, and (ii) 'in-betweens' belong exclusively and characteristically to what is continuous. In brief, I submit that the conflicts are resolved once it is shown that and how there is an important difference between what it is for the parts of a whole to be 'in between' one another and what it is for the limited to fall 'in between' the *limits* of a continuous thing. It should be evident that the major difficulty here is deeply connected with our earlier problem of conceiving how limits can belong to that which they limit without thereby becoming a *part* that might *compose* the whole. The connection between the two main claims hinges on certain issues in the characterization of the notion of indivisibles. Ultimately, the source of many of the uncertainties is to be found in the fact that, for Aristotle, continuity and indivisibility are not only not incompatible notions, they are intimately connected—but this is not something typically appreciated.

The import of this alliance between the continuous and the indivisible is clearest in Aristotle's careful treatments and transformations of the concept of 'oneness' (in which continuity always plays a pivotal role). Accordingly, it is perfectly natural that, having just defined continuity in the most general way (and before addressing continuity in terms of problems having to do with the non-composability of continua by indivisibles in Book VI), Aristotle turns his attention in the remaining chapters of Book V to questions concerning the 'oneness' of motion and the closely connected notion of contrariety.

Oneness and Contrariety

Physics V.4 begins with the claim that "Motion is said to be one in many ways, for we mean one in more than one way."[61] First, motion is one "generically according to the different categories to which it may be assigned,"[62] e.g., a change of place is generically one with every change of place, but a change in quality is generically different from a change of place. Next, it can be one specifically/formally whenever, "being one in genus, it is also within an indivisible species/form [ἀτόμω εἴδει]."[63] A still stronger or more restrictive sense is then proposed: "when it is one in its thinghood [οὐσία] or one in number [ἀριθμῶ]."[64] Aristotle claims that these distinctions are made clearer by considering more closely the three sorts of things that are involved in any motion: (1) the 'that which moves' (i.e., something moved, e.g., a human being or gold), (2) the 'that in which it moves' (e.g., in place or in affection (πάθει)), and (3) the 'that during which it moves', i.e., the time in which the motion occurs.[65] Concerning these three aspects of motion, he claims that while (2) makes something one in genus or in species, (1) makes the motion one "in the underlying reality [ἐν τῶ πράγματι]," and (3) "makes it contiguous [ἐχομένην]," it is the three together that make a motion one without qualification [ἁπλῶς]:

> For the 'that in which' [i.e., (2)] must be one and indivisible [ἄτομον], as the species/form [τὸ εἶδος], and 'that during which' [i.e., (3)] it takes

[61]Aristotle, *Aristotle's Physics*, 227b4-5.

[62]Ibid., 227b5-6.

[63]Ibid., 227b8-9.

[64]Ibid., 227b25.

[65]Ibid., 227b28-29.

> place, the time, must be one and not leave gaps, and the thing moved [i.e., (1)] must be one in a non-accidental way[...].[66]

This strongest sense of "unqualified" oneness of a motion, then, requires oneness in all three respects: in the genus (or species) of the motion, in the underlying πραγμα moved, and in the time. In elaborating on such strongly unified motions, he states that since every motion is continuous, then also the motion that is "one in its own right" must be continuous; but, he adds, this does not entail that every motion would become continuous with every other one, for that would require, by definition, that their extremities become one. But of certain things there simply are not extremities, and of others the extremities may differ in species/form. More generally, motions that are not one in either species/form or genus but are one in another way may be contiguous—"for someone might run and then at once fall ill of a fever, as also a torch relay race may be a contiguous change of place"[67]—but not continuous:

> Hence motions may be contiguous [ἐχομένη] and successive [ἐφεξῆς] in virtue of the time being continuous [τὸν χρόνον εἶναι συνεχῆ], but there can be continuity only in virtue of the motions being continuous, and this is when both become one at the extremities [ἓν τὸ ἔσχατον γένηται ἀμφοῖν]. Therefore, motion that is in an unqualified sense continuous and one must be the same in species/form, of one thing, and in one time. In one time, lest there be motionlessness in between [μεταξὺ] (for where there are gaps in motion there must be rest, and a motion that has rest in between will not be one but many, so that a motion interrupted by stationariness [στάσει] is not one or continuous, but it is so interrupted if there is a time in between). And though of a motion that is not one in species, even if it does not leave gaps, the time is indeed one, the motion is different in species, and so cannot really be one, for motion that is one must be one in species, though motion that is one in species is not necessarily one in an unqualified sense.[68]

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[66]Aristotle, *Aristotle's Physics*, 227b32-38.

[67]Ibid., 228a30-32.

[68]Ibid., 228a33-228b12.

Here, we see that while motions that are not one in genus or in form may be contiguous, they cannot be continuous—for precisely the reason that they are not suggenic. We note also that things may be contiguous and successive by virtue of continuity in the time; however, continuity in the time is not enough to make the motion and the πραγμα of the motion continuous in the most demanding of senses. Figure 2.1 (on the next page) shows how these types of motions are arranged. Notice how of the various combinations of the three kinds of oneness—namely, (not) one in species/form, (not) one in ousia, and (not) one in time—the top four combinations are those of the eight that are *continuous* and these correspond exactly to those where there is oneness in species/form/genus (while both the presence of oneness and its negation are represented with respect to oneness in ousia and in time). As long as a motion is one in species/form, it will be continuous, and it will fail to be continuous otherwise; however, only a motion that was one in all three ways would deserve to be called one and continuous in "an unqualified sense."

Because the issues of oneness are so deeply connected with the fundamental role of contrariety in changes, for all continuous changes were held to be between contraries, it is quite natural that the final chapters of Book V are devoted entirely to questions involving the nature of contrariety. However, the concern of those chapters is more with specific motions that are contrary, and so with more specialized questions having to do with those contraries. More importantly, as the very first quote from this section stated, motions are one in a number of ways on account of the fact that 'one' is said in many ways. So before returning to the *Physics* and its treatment of continuity and contrariety as it unfolds in later books, I will further untangle some of the connections between oneness, contrariety, and

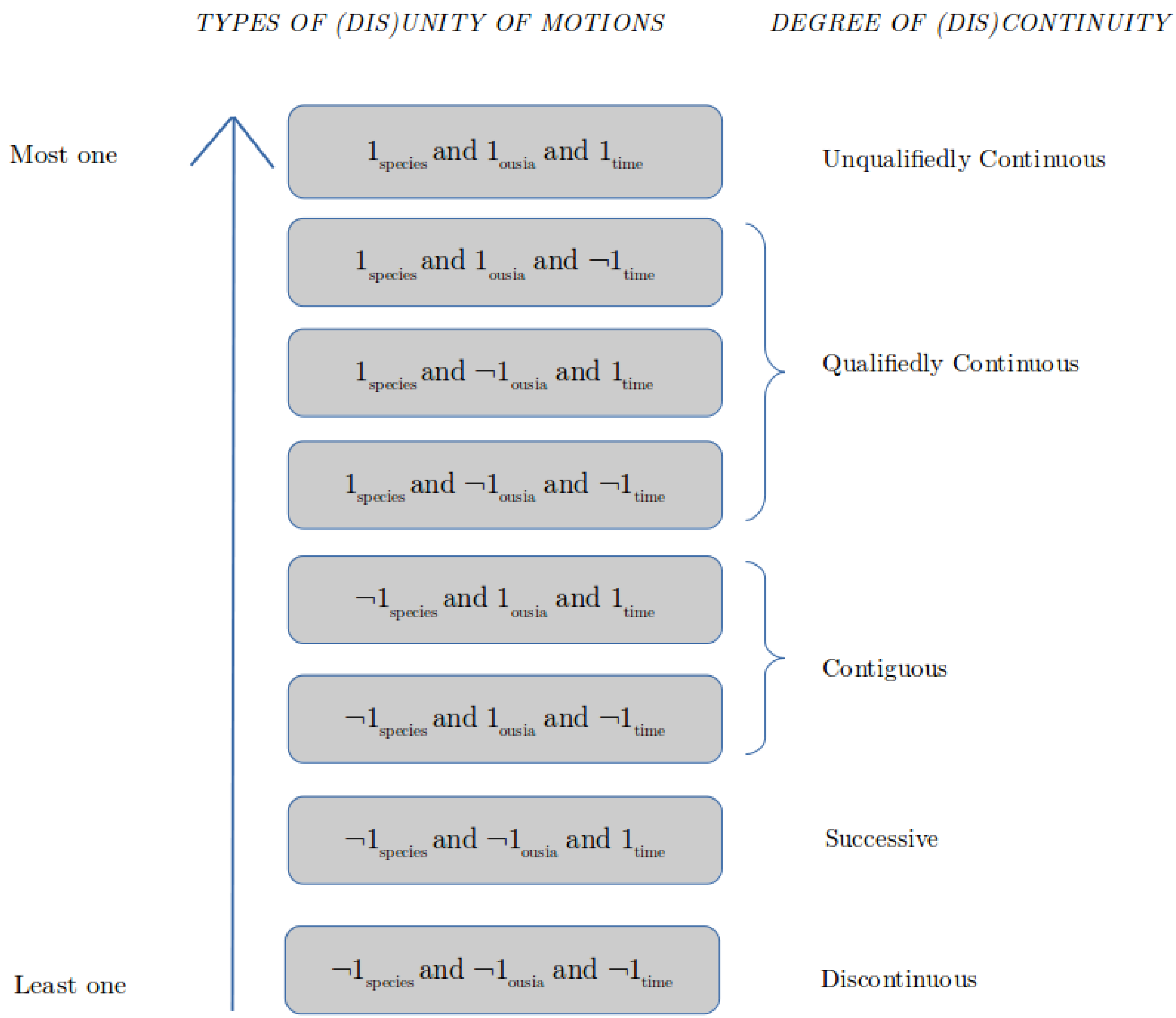

Figure 2.1: Oneness and Continuity of Motions

continuity through an interlude that covers the more general discussion of these matters in the *Metaphysics*.

Interlude: Metaphysics V and X

I made the claim earlier that it is mistaken to think that indivisibility and continuity are inherently incompatible notions. While this features prominently in the *Physics*, it is also made very explicit in Aristotle's discussions of oneness in the *Metaphysics*. In *Metaphysics*

V.6, there is a provisional discussion of the many ways in which 'oneness' is spoken of. Early on in this discussion, in discussing motion, we are already told that "what is called continuous is that of which the motion is one in its own right, and not capable of being otherwise, while the motion is one if it is indivisible and in an indivisible time."[69] V.6 goes on to outline some of the other senses of oneness, such as how "things are called one because what underlies them [τὸ ὑποκείμενον] is undifferentiated in form/species [τω εἴδει ιἶναι ἀδιάφορον]," or how things are called one "whose genus [τὸ γένος] is one, though they differ by opposite differentiae [διαφοραῖς], and these are all called one because the genus that underlies [τὸ γένος ἓν τὸ ὑποκείμενον] their differences is one."[70] To mention a few other senses of 'oneness': things are also generally called one when the articulation (ὁ λόγος) of their "'what it is' [τὸ τί ἠ̃ν εἶναι] is indivisible into any other one revealing the 'what is is' of some underlying reality [τί ἠ̃ν εἶναι τὸ πρᾶγμα]," even though "every articulation itself is divisible within itself [αὐ̃τὸς γὰρ καθ΄ αὐ̃τὸν πᾶς λόγος διαιρετός]."[71] Closely related to this last sense of 'one': those things are said to be one "of which the *thinking* [νόησις] is indivisible, which thinks the 'what it is' [τὸ τί ἠ̃ν εἶναι]."[72] This initial exploration of some of the ways 'oneness' "is said" inspires Aristotle to try to generalize from these initial senses, locating some feature each shares: "whatever does not have a division [μὴ ἔχει διαίρεσιν], insofar as it does not have it [ἠ̃ μὴ ἔ̃χει], is in that respect one."[73] This leads to the further observation that while most things

[69]Aristotle, *Metaphysics*, 1016a4-6.

[70]Ibid., 1016a17-18; a25-28.

[71]Ibid., 1016a34-37.

[72]Ibid., 1016a38-39.

[73]Ibid., 1016b1-3.

are said to be one in relation to something else, there are others that are "called one in the primary sense," and these are said to be four: that which is one in thinghood (οὖσια); one in continuity (συνεχεία); one in form/species (εἴδει); or one in articulation (ὁ λόγος).[74] After introducing these four principal senses of oneness, Aristotle mentions another key feature of 'oneness' in general:

> To be one is to be a source/principle [ἀρχη] for something to be a number; for the first measure [πρῶτον μέτρον] is a source, since that by which we first know each genus [γένους] of things is the first measure of it. So oneness is the source of what is knowable about each thing. But what is one is not the same in all genera [γένεσι]; for here it is the smallest musical interval, but there it is the vowel or consonant, and of weight it is a different thing, and of motion still something else. But what is one is always indivisible, whether in quantity [ποσω] or in form/species [εἴδει].[75]

I would like to emphasize a few things here, the full significance of which will emerge shortly: (1) that by which we first know each genus of things is the first measure of it; (2) such a first measure is a source or principle; (3) to be one is to be a source/principle; so (4) oneness is the source/principle of what is *knowable* about each thing; yet, at the same time, (5) what is one is not the same for all genera, though (6) what is one is always indivisible.

Already in this earlier foray into some of the ways in which oneness is "said," we can begin to appreciate that continuity is not simply opposed to indivisibility, but is in fact closely aligned with it. In returning to consider these matters of 'oneness' (τὸ ἓν) with more sustained and direct attention in Book X—and moving beyond the language of "those things are *called* one" to more direct statements about the nature of oneness—

[74] Aristotle, *Metaphysics*, 1016b4-10.

[75] Ibid., 1016b18-24.

Aristotle develops a number of very explicit connections between the various types of oneness (including continuity), the nature of contrariety, genera and suggenicity, and indivisibility. He begins Book X by offering a revised and more considered account of oneness, in which he says that, beyond the many ways 'oneness' is said, there are in fact four senses in which something is one "primarily and in its own right [τῶν πρώτων καὶ καθ' αὑτὰ]," as opposed to accidentally:

1. Things are one if they are continuous (συνεχὲς), either unqualifiedly/simply (ἁπλῶς), or in the highest degree (μάλιστά), by nature (φύσει), rather than just by contact (μὴ ἁφη) or by being tied together (μηδὲ δεσμῶ); and of these, those whose motion is more indivisible and more simple (μᾶλλον ἁπλη) are one to a higher degree and are prior.[76] (*Unity by "Natural" Continuity*)

2. Things are one if they are a whole (τὸ ὅλον) and have some shape or form (μορφὴν καὶ εἶδος); such a thing is said to be one most of all (μάλιστα) if it is by nature of this sort, and not by force (μὴ βία), as those things are that are so by means of glue or bolts or being tied together (ὅσπερ ὅσα κόλλη ἢ γόρφω ἢ συνδέσμω), *but rather has in itself that which is responsible for its being continuous* (ἔξει ἐν αὑτω τὸ αἴτιον αὑτω τοῦ συνεχὲς εἶναι). Something is of this sort if its motion is one and is indivisible in place and time. (*Unity by Wholeness*)

   The next two pertain to those things that are one in such a way that their λόγος is one, and those are precisely the things the thinking of which is indivisible, "and an act of thinking is indivisible if it is of something indivisible in form/species

[76] Aristotle, *Metaphysics*, 1052a17-23.

(εἴδει) or number (ἀριθμω),"[77] the former of which is aligned with universality and the latter with particularity:[78]

3. Indivisibility in number (ἀριθμω): a particular thing (καθ' ἕκαστον) is one by being numerically indivisible.[79] (*Numerical Unity*)
4. Indivisibility in form/species (εἴδει): what is responsible for the oneness of independent things (οὐσίαις).[80] (*Formal Unity*)

What these four "primary" senses of 'one' have in common is that "each of these is one by being indivisible, either with respect to motion, to an act of thinking, or in articulation."[81] In addition to observing that the very first sense of oneness is oneness by virtue of continuity—a sense that even embraces the second, to the extent that something is said to be a whole "most of all" precisely when it *has in itself that which is responsible for its being continuous*—we can also observe an unequivocal statement that *what is continuous is not incompatible with being indivisible* (at the very least, in one of the three senses of indivisible); still more, the continuous, as preeminently *one*, is not only not incompatible with indivisibility, but "is one *by being indivisible*." Aristotle reiterates this in the next passage, again forging a connection between the indivisible as that by which something is made one and the "first measure of each genus," claiming: *to be one is to be indivisible, and this most of all when it is the first*

[77] Aristotle, *Metaphysics*, 1052a29-31.

[78] Ibid., 1052a38-a40.

[79] Ibid., 1052a32-34.

[80] Ibid., 1052a35-37.

[81] Ibid., 1052b1-2.

measure of a genus (ἑκάστου γένους).[82] Concerning what it is to be a *measure*, Aristotle adds that "everywhere, we seek as a measure something that is one and indivisible," and "an exact measure" is attained "whenever it seems not to be possible to take away or add anything" (thereby forging a connection to "completion").[83] Not only is the measure of any thing that which is one[84]—something that will turn out to be relative to each given genus—but again we see Aristotle insist that "for this reason the one is indivisible, *because the first in each [genus] is indivisible*"; and on account of the differences in genera, "they are not all indivisible in the same way."[85] In other words, what is one will be indivisible precisely on account of the indivisibility of whatever is first in each genus, and the type of indivisibility itself will thereby be determined by the genus. However, in terms of the language we developed earlier (i.e., "internally"), it is equally the case that the measure will always be suggenic (συγγενὲς) with what it measures.[86]

In X.2, Aristotle narrows in on these issues, and pursues the question of the "thing-hood and the nature" (οὐσίαν καὶ τὴν φύσιν) of oneness/unity (τὸ ἕν), claiming that oneness itself cannot be some genus, and for basically the same reasons that "being" (τὸ ὄν) cannot

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[82] Aristotle, *Metaphysics*, 1052b16-19.

[83] Ibid., 1052b28-29; b29-30.

[84] Ibid., 1053a15.

[85] Ibid., 1053a15; a16-17. Aristotle also remarks on an important distinction at 1053b4-8, namely that what is one is what is indivisible "either simply or in the respect in which it is one," a distinction I think we may understand to mean indivisible *a-contextually* as opposed to indivisible with respect to the particular way the given genus has achieved its unity.

[86] Ibid., 1053a22-27. For future reference, we can call this the *Suggenicity of Measure-Measured.*

be a genus.[87] But this merely negative result does not yet address the other side of the matter: since among qualities there is a certain kind of oneness and some nature that is one, and similarly also among quantities, one must inquire more generally into what oneness is—"since it is not sufficient just to say that it is itself its own nature."[88] We have already seen this idea of not being able to take for granted how something achieves its unity—i.e., it is not sufficient just to say that oneness is "itself its own nature"—as well as the non-uniformity of oneness, spelled out in terms of the fact that what is one is not the same for all genera, but rather is proper to each genus. Significantly, it is precisely this non-uniformity of oneness that drives the discussion forward into a more thorough articulation of the notion of contrariety and indivisibility in relation to genera.

In first attempting to unpack oneness in its various guises in terms of its "thinghood," Aristotle introduces an issue that will become of great importance. He explains how, for instance, with colors, "that which is one is a color, such as white, and then the other colors clearly come to be out of this and black, black being the deprivation of white."[89] Accordingly, he famously says, "if beings were colors," they would be a certain kind of number—namely, a number of colors, and here "the one would be one something, namely whiteness."[90] In the same way, if beings were sounds or tones, "there would be a certain number, though in this case a number of quarter tones," and "oneness would be something whose thinghood

[87]This of course is not to be confused with the very different claim that *a genus is itself some sort of unity*—a claim Aristotle does in fact emphatically endorse.

[88]Aristotle, *Metaphysics*, 1053b26-29.

[89]Ibid., 1053b30-32.

[90]Ibid., 1053b33; b34.

would not be oneness but rather the quarter tone."[91] And similarly, if beings were "spoken utterances," beings would be a number of letters, and oneness would be the vowel. Moreover, the same sort of account applies to the other genera of things as well, e.g., among qualities and quantities and motions.[92] The moral of all this is that, across all of these cases, the oneness of the item in question would never be "oneness" in the abstract, as if this were some uniform property that could be grafted onto any objects, but rather some one item with a particular nature:

> It is clear, then, that oneness in every genus [ἅπαντι γένει] of things is some nature [τις φύσις], and that the nature of oneness is in no case just oneness itself [...].[93]

The remainder of Book X is largely occupied with determining how the idea that the nature and thinghood of "oneness" is not "just oneness itself," but is determined for each genus of things as some one nature, will unfold when we are no longer dealing with the special case of one *independent thing*, but when we come to consider pluralities unified into a whole, like qualities and other trans-individual realities that are irreducibly plural and are not one in the way one independent thing is one, yet that nonetheless achieve a certain (non-accidental and non-mental) unity.

In X.3, as if anticipating the listener's tendency to suppose that any plurality would automatically be barred from being a unity—i.e., that only what is one independent thing,

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[91] Aristotle, *Metaphysics*, 1053b36-1054a1.

[92] Ibid., 1054a3-5. To this he also adds that this account even applies in the special case of thinghood: "just as among colors one must look for one color that is itself what is one, so too in thinghood, one independent thing is oneness itself" (1054a13-14).

[93] Aristotle, *Aristotle*, 105410-12. In the special case of thinghood, oneness will be one independent thing.

or numerically one, could be *one*—Aristotle observes that "if pluralities are simply opposed to the one, impossible things follow."[94] In fact, "the one and pluralities are opposed *in many ways*."[95] Elaborating on this, Aristotle claims that

> one of these ways the one is opposed to plurality is as the indivisible to the divisible; for what is either divided or divisible is called a 'plurality', and what is indivisible or not divided is called 'one'. Now since there are four kinds of opposition, and since of the two opposites one of them is said to be opposed according to deprivation [κατὰ στέρησιν], one and plurality must be contraries [ἐναντία], but neither contradictories [ἀντιφασις] nor relatives [τὰ πρός τι λεγόμενα].[96]

The four sorts of opposition introduced here—contradictories, relatives, deprivation/possession, and contrariety—are technical terms in Aristotle's thinking. Throughout his corpus,[97] Aristotle holds that all changes are from either contradictories or contraries, and that all motions in their own right are from contraries—and the paradigmatic contrariety is that of complete deprivation and possession. The technical notion of contrariety is of particular importance to the account in the *Physics*, but while the idea is applied across all three primary changes—change in place, change in quality, change in quantity—the examples of contrariety Aristotle deploys more often than not involve changes in quality, i.e., alteration, such as changes in color from white to black via intermediates or from sweet to bitter, though occasionally he does consider quantities such as degrees of heat, density, etc.[98] In the present context,

[94]Aristotle, *Aristotle*, 1056b6.

[95]Ibid., 1054a20.

[96]Ibid., 1054a21-26.

[97]See, e.g., *Physics* 188b21-26; *Metaphysics* 1055b16-17; *De Caelo* 270a14-25; *De Generatione* 323b28-324a9.

[98]The case of quantities is more complicated. *Physics* V, 226a23-32, for instance, holds quantities to have contraries. But in the *Categories*, for instance, he says that "a quantity has no contrary. In the case of

Aristotle elaborates on the notion of *opposition*—in particular, while discussing the specific opposition between "sameness" and "otherness"—and introduces the very important concept of *difference* (διαφέρον). What differs can differ (διαφέρον διαφέρει) either in genus or in species; things differ in genus if the things do not have a common matter and if no generation (γένεσις) can exist from one to the other,[99] while things can differ in species if they fall under the same genus.[100] The notion of difference is a powerful tool in Aristotle's arsenal, designed to address not only how there are distinct genera, but how within the

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definite quantities it is clear that there is no contrary; there is, for example, no contrary to four foot or five foot" (5b12-14). I agree with Bogen (1992), when he claims that these can be reconciled by considering what Aristotle says more closely at *Physics* V, namely that for change in quantity, for which there are contraries according to this passage, "increase is change toward *complete* magnitude [εἰς τέλειον μέγεθος] and decrease [is change] away from this" (226a30-32). Bogen remarks:

> This passage [226a23-32] says that contraries figure in quantitative change and implies that without quantitative contraries there could be no change in quantity. I take 'complete magnitude' to be the largest size attainable by items of the same kind to which the changing individual belongs. And although Aristotle does not mention it, we can suppose there is a minimum as well as a maximum magnitude for items of any kind whose members can increase or decrease. On this supposition, our passage allows us to treat magnitudes in between the maximum and minimum as intermediates between contraries—relative of course to a given kind. (Bogen, "Change and Contrariety in Aristotle," 18-19.)

Bogen also argues, with respect to Aristotle's claim that things sharing the same capacities (δυνάμεις) support contrariety, that

> we can say that even though numbers *per se* have no contraries, the maximal and minimal amounts of measurable quantities, possessions, deprivations, etc., possible for things of any given natural kind are contraries to which numbers can be assigned. Any such numbers (and the intermediates which fall between them) are the measures of contraries—relative, of course, to the kinds for which they are maxima or minima. It follows that if something goes from the possession of one amount i, to the possession of another amount j, of some quantity, possession, privation, etc., this can be counted as a change to be accounted for in terms of contraries even though i and j are numbers. (ibid., 20.)

[99]Presumably, this lets us infer that if things are the same in genus, then they will have a common matter and there will be (at least in principle) γένεσις from one to the other.

[100]See Aristotle, *Metaphysics*, 1054b29-31.

confines of a given "same" genus—and at a "higher" level than any individual or numerical unification—the plural natures unified under that genus achieve determinacy. Given the trajectory of the discussion, the main purpose of discussing "difference" appears to be to observe that *contrariety is a certain kind of difference.*[101] In fact, contrariety is not just *any* kind of difference. From the fact that things that differ from one another can differ more and less (πλεῖον καὶ ἔλαττον), Aristotle infers that there must be "a certain kind of difference that is the greatest [μεγίστη], and this I call contrariety [ἐναντίωσιν]."[102] He claims that this is clear "by induction" (or "by examples") (ἐπαγωγης), using language and reasoning that should strongly remind the reader of issues from the *Posterior Analytics*:

1. Things that differ in genus (i) do not have a way to one another (οὐκ ἔχει ὁδὸν εἰς ἄλληλα), but (ii) hold apart too much (ἀπέχει πλέον) and cannot be compared (ἀσύμβλητα).[103] (*No Transit Between Genera; Incomparability of Genera*)

2. What is greatest *within* each genus is complete (τό γε μέγιστον ἐν ἑκάστω γένει τέλειον)—for what is greatest is that which cannot be exceeded (ὑπερβολή), and that is complete (τέλειον) outside of which (ἔξω) nothing can be found. This is true since "complete difference" (τελεία διαφορά) has an end (τέλος), and outside of the end there is nothing; for "in everything the end is an extreme and bounds

[101] Aristotle, *Metaphysics*, 1054b32-33.

[102] Ibid., 1055a3-5.

[103] Ibid., 1055a7-9. This claim should remind the reader of the *No Genus Crossing* principle from *Posterior Analytics*, discussed in the previous chapter. But just as there were certain qualified exceptions to this general rule as it manifested itself in the *Posterior Analytics*, *Physics* VII.4 discusses some restricted conditions under which motions can be "compared" with one another.

it all around [ἔσχατον γὰρ ἐν παντὶ καὶ περιέχει]."[104] (*"Internal" Completion and Closure of Each Genus*)

3. The complete difference is the maximal difference. (*Maximality—Relative to a Given Genus—of Complete Difference*)

4. As for things that differ in species/form: (i) their coming into being is from their contraries, as from extremes, and (ii) the interval between these extremes is the greatest. And so, the interval between contraries is also the greatest.[105] Thus, for each species/form, necessarily sharing a genus, it further follows that

5. Contrariety is the maximal (μεγίστη) difference, from which it is immediate that

6. *Contrariety is the Complete Difference*,[106] for "there is nothing outside" this interval.

Unlike distinct genera, incomparable among themselves, the extreme contraries of a single genus—as the maximal and complete difference within that given genus—will involve comparability among all that falls in between its boundaries. On account of the suggenicity of the measure and the measured, it will be possible to impose a metric on the interval formed from one extreme to the other, according to which a comparison between what will fall between the two extremes may be effected. Distinct genera, by contrast, are strictly incomparable or incommensurable, for it is not possible for them to be measured by the same measure, and so to be positioned on the same continuum, according to which one might evaluate or compare

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[104]Aristotle, *Metaphysics*, 1055a11-14.

[105]Ibid., 1055a10-11.

[106]Ibid., 1055a16.

the "nearness" of the one to the other. This is reinforced by the fact that the complete difference is a difference "beyond which there is nothing."[107] Being able to find a measure, a standard of comparison, is by no means something arbitrary for Aristotle, or reducible to a question of the practices of human minds: it is only if two things already belong to the same genus that there may even be a common measure. No matter how hard you try, you will not be able to find a measure to determine whether a singer's high note is 'sharper' than your friend's wit, for the "underlying reality" in these cases involve distinct genera. On the other hand, Aristotle is clear that there is indeed an exact measure by means of which, within a given genus such as that represented by color or tone, colors and tones can be compared and regarded as more or less "distant" from one another; and the determinations of such a measure are ultimately provided by the *maximally extreme contraries* of the interval in question.

In addition to asserting that "things in the same genus that differ most are contraries,"[108] Aristotle adds that, in accordance with the demand that the incommensurability of certain natures derive from the difference in the recipients (δεκτικά), the things in the same recipient that differ most are contraries (since the same material belongs to contraries).[109] Moreover, things subject to the same power/potency (τὰ ὑπὸ τὴν αὐτὴν δύναμιν) that differ most are contraries.[110] Because what is greatest in each genus is contrariety; because there

[107]Precisely this feature is what has led me to refer, throughout this chapter, to the completion of a genus as an "internal completion."

[108]Aristotle, *Metaphysics*, 1055a28.

[109]Ibid., 1055a29-30.

[110]Ibid., 1055a31-32. Applied to motions, this last claim stipulates that the changes something can bring about, either in itself or in another, and the changes it can undergo, depend on the characteristic potencies

cannot be anything more extreme than the extremes of complete contrariety; and because there cannot be more than two maximal extremes to any interval,[111] Aristotle now claims that "it is not possible to be contrary to more than one thing."[112] In other words, since there cannot be more than two maximal extremes for any interval, every contrary is contrary to one and only one thing (i.e., contraries are *unique*).[113]

While it might almost seem unnecessary to state, given how frequently it has already been assumed, it is important to realize that *contraries are the extremes from which change proceeds*.[114] Building on this notion of the maximal difference found in complete contrariety, Aristotle notes that the primary sort of contrariety, from which the rest are derived, is that of possessing (ἕξις) and deprivation (στέρησίς)—not just any deprivation, but complete (τελεία) deprivation.[115] While deprivation is initially said to be a certain kind of contradiction or incapacity (ἀντίφασίς τις ἢ ἀδυναμία), this claim is qualified: for deprivation can also be "taken together with the material receptive of what is lacking," so while "there is no in-

or powers (δυνάμεις) belonging to things of that kind; this should be compared to the claim, cited earlier, that all states by virtue of which things are *impassive* (απαθη) to change are called potencies (δυνάμεις) (where we further know that continuity is characterized by its components being impassive to one another).

[111]Aristotle, *Metaphysics*, 1055a21-22.

[112]Ibid., 1055a20.

[113]One might wonder, then, why the contrary of white is black, and not red, blue, green, etc. However, it is in fact precisely the notion of contrariety as the *complete* and *maximal* difference that is meant to account for this uniqueness, while at the same time allowing for the existence of "lesser" contraries intermediate to the extremal ones.

[114]Aristotle, *Metaphysics*, 1055b17-18.

[115]Ibid., 1055a34-38. Why would Aristotle bother to specify that the deprivation in this case is "complete" if it were not for the fact that he understands possession and privation to admit of degrees, i.e., continuous variation?

between in contradiction, in some deprivations there is."[116] Now, every contrariety ultimately has a deprivation as one of the two contraries, but this is something that is also not uniform (i.e., is something that depends on the given genus). For instance, while "inequality is the deprivation of equality, and unlikeness of likeness, and vice of virtue," these differ in that, e.g., something can be deprived at some time or in some respect, or in the decisive respect, or in every respect.[117] And these are some of the reasons why "there is an in-between for the one sort, and it is possible that a human being be neither good nor bad, but for the other sort there is not, but a thing has to be either odd or even."[118] Moreover, while some contrarieties have a definite underlying subject, others do not.[119]

In elaborating on the particular case of "equality" and "inequality" in relation to contrariety, Aristotle claims that

> the equal appears to be an intermediate between the great and the small, but neither does it seem that any contrary thing [as a complete contrary] is an in-between, nor is it possible from its definition, for it would not be a complete contrary if it were between things, but a contrary instead always has something between itself and its contrary.[120]

This is a very basic, but important, distinction: a *complete contrary* cannot *itself* be in-between anything (since it supplies the measure according to which there is any 'betweenness'), yet a contrary does always have something in between it and its contrary. We could

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[116]Aristotle, *Metaphysics*, 1055b7-8; b9-10.

[117]Ibid., 1055b18-25.

[118]Ibid., 1055b25-28.

[119]Ibid., 1055b28-29.

[120]Ibid., 1056a13-16.

call this the *Betweenness of Contraries*: a contrary, in particular that formed by the complete difference, always has something between itself and its contrary. As for 'the equal', then, this is "that which is neither great nor small but can by nature be great or small, and it is opposed to both as a negation deprived of both, and this is why it is between them"; something similar occurs for "that which is neither good nor bad [and] is so opposed to both good and bad."[121] But the "joint negation of opposites" that occurs here is something that does not occur in the same way in all cases:

> [Rather,] the joint negation of opposites is present for things which are of such a nature that there is some sort of in-between and interval. But between things of the other sort [i.e., things not of such a nature that there is an in-between and an interval] there is not a difference, since the joint negations are of things of which each is in a genus other than [the genus] of the other, and so the underlying subject is not one.[122]

This passage again exposes how there can be no in-between between distinct genera, while there are indeed in-betweens precisely in the interval spanned by each contrary, in particular between the complete difference, *within* a given genus. Within a given genus, by virtue of complete contrariety, there are natures that will support an in-between and a "joint negation" of the extremes, enabling intermediary states.[123] Chapter 7 of Book X offers a clear restatement of these points, bringing together much of the preceding topics discussed in this interlude, as well as beginning to draw us back into our main problem:

[121] Aristotle, *Metaphysics*, 1056a24-29.

[122] Ibid., 1056a24-b3.

[123] It is difficult to see how the law of excluded middle cannot fail whenever one is dealing with changes that proceed along the intermediate levels of a closed genus, between the complete contraries that form the extremes of that genus. It is still more difficult to deny that Aristotle does indeed countenance the existence of such things, and does indeed acknowledge that the logic applied to contradictories does not apply in such cases.

> Since contraries admit of having something in-between, *and some do indeed have it*, it is necessary for what is in-between to be made out of [ἐκ] the contraries. For all in-between things are in the same genus [ἐν τω αὑτω γένει] as the things they are between. For we speak of as in-between those things into which something that changes must change first (for instance, if one were to change from the highest to the lowest tone of a chord by steps of the least interval, one would come first to the in-between sound, or in colors if one were to change from white to black, one would come to red or gray before black, and similarly in other cases), but to change from one genus [ἐξ ἄλλου γένους] into another genus [εἰς ἄλλο γένος] is not possible other than accidentally, such as from a color into a shape. Therefore it is necessary for the in-between things themselves to be in the same genus [ἐν τω αὑτω γένει] and in the same one [αὑτοῖς] as the things they are between.[124]

Here we observe very explicit statements that contraries admit of in-betweens and that all in-between things are necessarily in the same genus as one another and as the things they are between (*Generic Unity of In-Betweens and their Extremities*). This latter idea should be compared to the *Suggenicity of Measure-Measured* from before—the measure is always suggenic with what it measures—which was a claim that concerned *measured* beings. Notice that while these two claims are very closely related, the *Generic Unity of In-Betweens and their Extremities* is a stronger, and ontologically more fundamental, claim about *beings* and the changes those beings support. I should also point out that Aristotle's current definition of in-between is substantially the same as that offered in the earlier determination of continuity from *Physics* V.3; in this connection, I should also remind the reader that in that chapter, in-between things were held to belong characteristically to what was continuous.

In what follows, Aristotle comes to the heart of our main issue:

> But if in-between things [τὰ μεταξύ] are in the same genus [γένει] as the things they are between, as has been shown, and are between contraries,

[124]Aristotle, *Metaphysics*, 1057a19-29, my emphasis.

> it is necessary that they be composed out of [συγκεῖσθαι ἐκ] these contraries. For there will be either some genus [τι γένος] that includes them, or none.[...] Now contraries are not composed of one another [ἀσύνθετα ἐξ ἀλλήλων], and are therefore sources/principles, while the things between them are either all composed of them, or none. But something comes to be out of the contraries [ἐκ τῶν ἐναντίων γίγνεταί τι], in such a way that a change will be into this before it is into the other contrary, for it will be more of one of them and less of the other. Therefore this also is in-between the contraries. And therefore all the other in-between things are composite [πάντα σύνθετα τὰ μεταξύ], for a thing that is composed of the more and the less is in some way derived from [πως ἐξ] those things with respect to which it is said to be more and less. And since there are no other things prior to them that are the same in kind [ὁμογενη] as the contraries, all the in-between things would be derived from the contraries, so that also all the lower kinds [τὰ κάτω πάντα], both contraries and in-between things, would be derived from the first contraries. That, then, all in-between things are in the same genus [ταὐτω γένει] as, and are between, contraries, and are all composed of those contraries [σύγκειται ἐκ τῶν ἐναντίων πάντα], is clear.[125]

Among other highly relevant claims, this passages tells us that *contraries are not composed of one another, and are therefore principles* (*Contraries are Non-synthetic; thus, Contraries are Archic*). As principles, any change will thus come to be out of these; and so, that which is in-between contraries—whether as "lesser" (subordinated) contraries or whatever is between any of the contraries—is not only necessarily in the same genus as the extreme contraries, but is necessarily made out of (σύγκειται ἐκ) those contraries. In other words, there is *Composition of all In-betweens from their Archic Contraries.* These two main claims represent a distinction vital to everything we have been discussing. The *maximal* contraries, that give the complete difference within a given genus, are not composed of one another. Moreover, as such, they are *principles* and are themselves *indivisible.* Still more, they are not

[125] Aristotle, *Metaphysics*, 1057b2-35.

themselves in between anything, for “there is nothing outside them”; and so such indivisible principles are incomparable (in relation to one another). However, whatever falls *in between* these maximal contraries—“and some do indeed have such in-betweens”—and so belongs to the same genus, is necesssarily “from” or “made out of” those contraries, in the sense that the latter act as a source/principle and measure for all the intermediate states, for we saw that, ultimately, their being amounts to an incomplete “possession” or “deprivation” of one of the contraries, against which such intermediate states are measured and admit comparison.

Having registered these important propositions regarding contraries and the previous vital distinction, Aristotle’s observation at the beginning of the next chapter (X.8), where he now deals with the special question of specific differentiation of genera in terms of “divisibility” and “indivisibles,” is entirely unsurprising. Being other in species is said to be just “the contrariety that things in the same genus, that are indivisible, have,” for “in divisions contrarieties come about also in the in-between things before one comes to the indivisible ones.” Division primarily concerns non-maximal contraries and that which falls between those contraries. As such, since such “secondary” contraries cannot provide the *complete difference*, which stands as the principle and measure from which all other contraries and their in-betweens are derived and measured, the unity of the underlying nature itself and its true measure must remain still other than the division would suggest. However, specific difference is something like the complete difference in a genus, and as such it is indivisible, or what terminates any division in the underlying genus. We are also now in a better position to appreciate the full import and seriousness of Aristotle’s appeal, in Chapter 9 of Book X, to the *aporia* concerning “why one pair of contraries makes things different in species but another does not, as footed and winged do while white and black do not,"[126] in response to which Aristotle tentatively asks: "Is it that the one pair are attributes fitting the nature of the genus while the other are less so?"[127] We will look more closely at these questions in the subsequent chapter.

Aside from looking ahead to the next chapter, I mention such things because it allows us to appreciate that all these issues involving contrariety and the nature of the internal continuity of genera and their external discreteness bear on such fundamental issues as the nature of essential definition. Without a prior understanding of how Aristotle is especially concerned in this book and beyond to establish that and how the genus must be at once a unity and capable of supporting in-betweens (and so division)—and how there is a distinction between incomplete divisions and the complete difference—such claims are mostly incoherent. However, following the train of the argument from the first propositions to the last, it becomes more apparent how all these pieces must fit together. The manner in which a genus is at once a unity and a continuum capable of supporting intermediate states—i.e., at once a 'one' and a 'plurality'—emerges as nothing other than the way that the indivisible limits or principles of a genus act to enable certain in-betweens and to unify this plurality in providing the "complete difference" against which all the possible intermediary changes are to be measured and compared among themselves. The more special questions concerning the relation between species and genus and how we know "why one pair of contraries makes things different in species but another does not" can now be understood in a more general

[126] Aristotle, *Metaphysics*, 1058a35-37.

[127] Ibid., 1058a38-b1.

light as a consequence of deeper commitments concerning the structural relations between indivisibles, genera, and continua.

Genus, in the more flexible (but still precise) sense developed above, captures what is shared by things that differ in certain respects. It is not primarily a matter of predicational commonalities, but the existence of unifications of trans-individual or irreducibly plural natures. While there are importantly *not* intermediates *between distinct genera* (for distinct genera are characteristically incomparable and so discretized); *within each given genus*, spanned by the interval formed from the extreme contraries, there must be in-betweens and, as such, a genus can be understood to support continuous variation. For ease of reference, we have spoken of the former way in which distinct genera emerge as characteristically incomparable and discontinuous in relation to one another as the "external discreteness" of genera. The latter way in which each genus is closed and completed by its limits in such a way that, within the interval spanned by the complete contraries of that genus, it supports in-betweens, has been called the "internal continuity" of genera.[128]

The degree to which most commentators have put disproportionate emphasis on the *Uniqueness of Contraries* principle and the *No Transit Between Genera; Incomparability of Genera* principle, has often misled readers into unnecessary "developmental" accounts of the nature of genus. But no "development" is needed to account for Aristotle's theory of genera once one has accounted for the fact that genera are *made* continuous with respect to all that falls between these limits and discontinuous with respect to one another by the

[128]If desired, one could easily make this notion of the *interior* (and, by extension, exterior) of a continuous thing more precise: say that any y between the limits of some Z is in the interior of Z if the parts or degrees of y are those parts or degrees of Z that are not contiguous with any parts or degrees other than the parts or degrees of Z.

action of the indivisible limits. *Distinct genera* are not only *externally* incomparable and discrete; but, as completed and bounded by their limits, genera are finite. However, in the determinate interval formed between the extreme contraries of a *given genus*, that genus supports continuous variation and so also comparable in-betweens. In supporting such in-betweens, a connection between continuity and infinite divisibility is forged. However, it should be clear how this connection is only part of the story: for, the primary characteristic of such in-betweens is not that they are infinitely divisible, but that they are *unified* into a single genus. Moreover, we saw that such in-betweens are even said to be *composed of* the extreme contraries, which act as principles and ends for them, in the sense that their "intermediate" status is precisely and entirely determined by their share or "possession" of one of the two contraries.

The above section should begin to indicate that the external discreteness and the internal continuity of genera cannot be understood in isolation from one other, and that one should not seek to highlight one aspect at the expense of the other. Both the external discreteness and the internal continuity of genera are the result of the characteristic action of the limit and principle, itself indivisible, by which a trans-individual generic plurality is at once made a determinate, incomparable, and unique reality (by virtue of its closure and completion through the limit in the maximal and unique pair of contraries)—and so a *one*—and one that supports, within a determinate and finite range carved out for it by the former action, a *continuum* of intermediates (comparable among themselves). Moreover, this account makes it more evident how a continuum can at once be infinitely divisible and uniquely characterized by its indivisible limits. Having clarified this, we are now in a strong position to return to the *Physics* in order to more directly address the divisibility of continua and the way they are "non-composable" from indivisibles.

Continuity Refined

Physics VI

In the previous section we saw, among other things, that indivisibility and oneness were said in many ways. While oneness is, importantly, not the same for every genus, what is one is always indivisible (in some sense). Aristotle's more considered view held that oneness in the most important, non-accidental, sense was fourfold: continuous; whole; numerically one; formally/specifically one. However, there were certain general features belonging to each of these four decisive senses of oneness-by-indivisibility, general features that made what is one be a source or principle:

> Most of all [to be one by being indivisible] is to be the first measure of each genus [ἑκάστου γένους], and in the most governing sense, of the class of things with quantity, for it is from this that it has been extended to the others.[129]

Finally, we also saw in the previous section that the differentiation of genera by means of the extreme or maximal contraries required that the genera of which the contraries were the differentiation (1) support in-betweens and the continuous variation thereby implied, and that they (2) be *completed* (and bound "at the extremities") by this maximal difference. We saw, moreover, that the contraries—as the indivisible principle of the continua the boundaries of which they define—are not composed of *one another*; but at the same time, that whatever

[129] Aristotle, *Metaphysics*, 1052b16-19.

is in-between contraries is both necessarily in the same genus as the extreme contraries (thus ensuring the suggenicity condition), and necessarily *made of* (σύγκειται ἐκ) those contraries.

Picking up where we left off in the *Physics*, it might seem strange, then, that Aristotle begins Book VI with the claim that if we assume the definitions of continuity, contiguity, and in succession, as determined in the previous book, it follows that "it is impossible for what is continuous to be composed of [or just: from] indivisibles [ἀδύνατον ἐξ ἀδιαιρέτων εἶναί τι συνεχές], for instance, a line cannot be made of points, if the line is continuous and the point indivisible" (231a24-26). However, this is not only not inconsistent with what we observed in the previous section regarding the relations between genera, continua, and indivisibles, but the ideas developed in the previous sections will help to illuminate the overall outcome of Book VI's treatment of this question.

We already know that indivisibility is importantly said in more than one way. But Book VI will take this still further, showing that what holds for *alterations*, or changes in quality, is different from other sorts of changes, and different precisely in terms of the role of indivisibles in this sphere. Typically commentators seem to miss this, either focusing exclusively on the main question of this book only as it applies to *quantitative* changes, ignoring the fact that Aristotle is addressing other changes as well; or, if they do acknowledge that the discussion applies to qualitative changes as well, it is not adequately appreciated that, precisely in connection with this issue of the role of indivisibles in the composition of continua, *qualitative change does not behave like the other changes.* And it is in this connection that we can better understand the important distinction between what it is to be a part composing a whole and what it is to be a limit that completes that of which it is the limit, but does not compose it as a part might.

Book VI is largely concerned with the nature of "in-betweenness." The problem bears directly on the fact that, e.g., something changing color must, while changing, characteristically fail to be completely either the initial extreme color from which the change set out or the final color. But it seems that this would only be possible if there is no end to division in the thing continuously changing, and if, strictly speaking, there is no *first* part, position, or instant at which the thing that changes first changes. And since motion in general must be continuous, Aristotle will hold that motions cannot be divided into indivisibles, nor can the continuous things that support the change be thus divided—from which he concludes, assuming that what can be thus divided can be thus composed, that whatever is continuous is not composed of indivisibles.

Aristotle's initial account of why continua cannot be composed of indivisibles begins by observing that the extremities (ἔσχατα) of two points cannot be one, since "of an indivisible there can be no extremity distinct from some other part [μόριον],"[130] nor can the extremities be together (ἅμα), since whatever is without parts cannot have an extremity, the extremity and that of which it is the extremity necessarily being distinct. Assuming a continuous thing were made of points, these would have to be either continuous or touching (ἁπτομένας) one another—"and the same argument applies to everything indivisible."[131] But, Aristotle observes, these points could not be continuous, and regarding their ability to "touch," one thing can touch another in one of three ways: whole to whole (ὅλον ὅλου), part to part (μέρος μέρους), or part to whole. However, anything indivisible is necessarily

[130] Aristotle, *Aristotle's Physics*, 231a29.

[131] Ibid., 231a36.

without parts (ἀμερὲς), so the only option is that it touches "whole to whole." But anything touching in this way could not be continuous, for the following reason:[132]

Proposition 2.0.0.1. The continuous has one part distinct from another (τὸ μὲν ἄλλο τὸ δ ἄλλο μέρος) and is thus divided into things that are different and separated in place (οὕτως ἕτερα καὶ τόπω κεχερισμένα).

According to the main argument, then, it could not even happen that

> a point was in succession [ἐφεξῆς] to a point, or a now to a now in such a way that the length or the time would be composed of them; for things in succession are those things of which nothing of the same genus is between [ἐφεξῆς μὲν γάρ ἐστιν ὧν μηθέν ἐστι μεταξὺ συγγενές], but between points there is always a line and between nows always a time. Further, it would be divisible into indivisibles [εἰς ἀδιαίρετα], if each thing is divisible into those things of which it is composed, but it was seen that no continuous thing is divisible into things without parts [οὐθὲν ἦν τῶν σενεχῶν εἰς ἀμερῆ διαιρετόν]. Nor is it possible for there to be anything of another genus in between [ἄλλο δὲ γένος οὐχ οἷόν τ᾽ εἶναι μεταξὺ], for it would be either indivisible or divisible, and if divisible, either into indivisibles or into the always divisible, in which case it is continuous. It is clear that everything continuous is always divisible into divisibles [πᾶν συνεχὲς διαιρετὸν εἰς αἰεὶ διαιρετά], for if it were divided into indivisibles, an indivisible would be touching [ἁπτόμενον] an indivisible, since the extremity of continuous things is one and touching.[133]

In the interests both of clarifying the argument and of revealing some of these more fundamental commitments on which it rests, I will offer a reconstruction of the argument presented in this passage. Let us first set out the major *explicit* assumptions Aristotle will rely on throughout the argument:

[132] I mark off a few claims as "propositions" in this section, since the main argument will make repeated reference and appeal to these ideas.

[133] Aristotle, *Aristotle's Physics*, 231b7-18.

Proposition 2.0.0.2. That which has no parts can have no extremity, where the extremity and that of which it is the extremity are distinct.

Proposition 2.0.0.3. One thing can touch another only whole to whole, part to part, or part to whole.

Proposition 2.0.0.4. What is touching whole to whole would not be continuous. (See 2.0.0.1.)

Theorem 2.0.1. Everything continuous is always divisible into divisibles.

Proof. Let X be something continuous. Assume, for the sake of contradiction, that it is composed of indivisibles.[134] Take X_1 and X_2 to be two (unspecified) indivisibles that together "compose" X. A first pivotal assumption made by Aristotle is that since the whole X is continuous, X_1 and X_2 must be either continuous with one another, touching one another, or in succession to one another. In other words: if what is continuous can be composed from disjoint indivisible parts, then those parts must be related in at least one of the three ways described in *Physics* V.3.[135] For ease of reference, call this assumption *Distribution of Continuity* (since the requirements at the level of the whole distribute to the parts.) We consider the three cases separately.

[134]Aristotle at first speaks of points (and later adds nows), but he then claims that the argument applies to "everything indivisible" and, later on, that the theorem concerns "everything continuous." In terms of the former claim, more precisely, he claims that the notion that indivisibles composing continuous things would have to be continuous with one another or touching one another applies to everything indivisible. But this is the assumption that structures the entire proof. Accordingly, in what follows, we keep things abstract.

[135]I say "disjoint" because they cannot be overlapping, otherwise they would have overlapping parts; but, as indivisible, they do not have parts.

Case 1. Let X_1 and X_2 be *continuous* with one another. Then X_1 and X_2 must have extremities that are touching and one (by 2.0.7, the definition of continuity). But if an indivisible has an extremity at all, by 2.0.0.2 that extremity cannot be distinct from some other part of it, for indivisibles have no parts. Since continuity (see 2.0.7 and 2.0.0.1) requires that the extremity and that of which it is the extremity be distinct, the extremities of X_1 and X_2 cannot be continuous with one another. By the *Distribution of Continuity*, then, X cannot be continuous. But X was assumed continuous. Contradiction.

Case 2. So let X_1 and X_2 be *touching* instead. Since whatever is indivisible is without parts, and since (by 2.0.0.3) one thing can touch another only whole to whole, part to whole, or part to part, it follows that as indivisibles, X_1 and X_2 could only touch whole to whole. But then X_1 and X_2 are not separated in place. By 2.0.0.1, anything the parts of which are not different and not separated in place cannot be continuous. Thus X_1 and X_2 *together* could not make a continuous X, for anything continuous must have distinct parts that are different and separated in place. But, again, X was assumed continuous. Contradiction.

Case 3. Let X_1 and X_2 be in succession.[136] By 2.0.5 (the definition of "in succession"), what is in succession is such that (1) it follows after the ἀρχὴν ; (2) there must be *nothing of the same genus* between it and that which it succeeds (though nothing prevents there being something that is not of the same genus in-between); and (3) successive things are ordered in terms of priority-posteriority, i.e., succession is anti-symmetric. In this demonstration, Aristotle will mainly make use of (2), i.e., that things in succession have nothing *of the same genus* between them. At this point, the argument becomes very abridged, and without

[136]In this part of the demonstration, we see the importance of the *suggencity* requirement.

spelling out the tacit assumptions to which Aristotle appeals, one is bound to find the argument deficient. Aristotle simply declares that if X is a line, with X_1 and X_2 as points, or if X is a time, X_1 and X_2 as nows, we are done—for "between points there is always a line and between nows always a time."[137] We can reconstruct this argument in more general terms. We recall that X_1 and X_2 are indivisibles composing the continuous whole X. We are trying to show that they can be in succession. Assuming they are in succession, and using the fact that things in succession are those things for which nothing of the same genus is in-between, there could be nothing of the same genus as X_1 and X_2 between them. But since X_1 and X_2 together compose X; and since X is a continuous whole; and since between the limits of a continuous whole, there are always in-betweens that belong to the same genus as the extremities; it follows that, as X_1 and X_2 are extremities, there must be something of the same genus as X_1 and X_2 that comes between them, call it X_3. But the existence of X_3 contradicts the assumption that X_1 and X_2 are in succession.[138]

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[137]At first, this seems very strange. For, one might object, a line does not seem to be "of the same genus" as a point. But just as place was not a body, yet, as the limit of a body, remained of the same genus as body, this objection fails to appreciate that, in a very particular way at least, Aristotle would speak of a point and a line as of the same genus. Another objection might say that given that Aristotle is aiming to show that, with respect to e.g., continuous lines and indivisible points, points cannot *compose* lines, it is not at all clear how the inference from "between any points there is a line" to "between any two points there is a point" is supposed to go through. However, such an objection again fails to appreciate certain subtleties in Aristotle's account of the relation between limit and the limited. Finally, one might also have the suspicion that Aristotle's argument simply assumes what it set out to prove—namely that continua are always further divisible. But closer inspection reveals that the fact that continua are divisible in one way is just a consequence of applying his stronger definition of continua from Book V.

[138]The careful reader will object: 'but I thought you were insisting, earlier, that whatever is continuous is necessarily in succession?! But this argument only goes through if you assume that what is continuous is *not* in succession!' But here we are in fact using the continuity of the *whole* X to show that—*were X to be composed of indivisibles*—those indivisible parts could not be arranged in succession. Once Aristotle has shown that continua are not in fact composed of indivisibles, there will be no problem showing that such continua do indeed have parts that are, in a way, also arranged in succession.

Remark. At this point, Aristotle takes things in a somewhat unexpected direction. He adds that it is not even possible for there to be anything of *another* genus in between X_1 and X_2. By the definition of succession—a definition that explicitly allowed for there to be something of another genus in between—this addition to the demonstration would seem to be unnecessary, and even to indicate that Aristotle has forgotten his own definition. But let us consider this more carefully.

As before, we let X be something continuous, with X_1 and X_2 as the two indivisibles together "composing" X. Aristotle now explicitly assumes that "each thing is divisible into those things of which it is composed." For ease of reference, call this: *Division and Composition are Inverse Operations.* By the latter principle, as X was assumed to be *composed* of the indivisibles X_1 and X_2, it is thereby *divisible* into the same indivisibles X_1 and X_2. But, again, by definition, indivisibles are without parts, and (from before) we know that no continuous thing is divisible into things without parts. It is precisely here that Aristotle claims that it is not even possible for there to be anything of *another* genus in between X_1 and X_2. The context is rather important. The context suggests, and the cogency of the demonstration requires, that whatever is both between X_1 and X_2 and of another genus than those two, also contributes (specifically by virtue of *Division and Composition are Inverse Operations*) to the *composition* of X. Otherwise, it is not only difficult to see why he would even spend time on this alternative (given the definition of succession), but he would in fact contradict all those other occasions where he explicitly holds that nothing prevents there being something of another genus in between (in which contexts he must certainly mean that they can be said to be in-between, but not that they thereby compose the whole the limits of which they are between). So let us assume that he is indeed specifically addressing the possibility that something of another genus is not just between X_1 and X_2 in an innocuous or weak sense, but that, as in between the limits of the whole X, it thereby divides X and so contributes to the composition of X.

Case 4. Let Y_3 be such a thing between X_1 and X_2 ('Y' to designate that it is of another genus than the X_i's, and Y_3 to designate that it introduces a third). But Y_3 must itself be either indivisible or divisible. We consider these cases separately.

Case 4.1. Let Y_3 be divisible. Then it is either divisible into indivisibles or into what is always divisible. If the latter, then Y_3 is (weakly) continuous.[139] Aristotle seems to go directly from the weak continuity of Y_3 to the (strong) continuity (see 2.0.7) of X_1, X_2, and Y_3—which, of course, is "impossible" for precisely the reason that such continuity required that the parts of the whole be suggenic. Something in one genus (Y) cannot, of course, be suggenic with something in another genus (X)! If we do not assume that this is what Aristotle has in mind here, then the argument is not only fallacious, but all it would have shown is the continuity of X_1, X_2, and Y_3 together, and this in turn would imply the continuity of X. But that was already assumed at the outset, so there would be no impossibility. As for whether or not he is justified in moving from the weak continuity of Y_3 to the continuity of X_1, X_2, and Y_3, the real question is whether or not he even needs to assume this, or is indeed assuming it. After all, if the continuity of a whole requires suggenicity of the parts and the extremities, whether we take Y_3 to be a part of the whole or another extremity, the addition of a weakly continuous Y_3 to X can only compromise the continuity of X.

[139] I say "weakly" continuous to distinguish infinite divisibility from the official and stronger definition of continuity from Book V. Unless I specify that I mean weak continuity, by continuity I always mean the continuity of 2.0.7.

So let Y_3 be divisible into indivisibles instead. Then, just as for X_1 and X_2, the indivisibles Y_1 and Y_2 into which Y_3 is divided cannot be continuous with one another, touching one another, or in succession. And so, assuming Y_3 also contributes to the composition of X, by the *Distribution of Continuity* principle, X will not be continuous, which contradicts the assumption.

Case 4.2. Let Y_3 be indivisible. By the same reasoning as the rest of the proof, this will not be consistent with the assumed continuity of X.

□

Immediately following this argument, Aristotle observes that the same reasoning applies equally to magnitude, time, and motion—either all are composed of indivisibles (ἐξ ἀδιαιρέτων) and are divisible into indivisibles (διαιρεῖσθαι εἰς ἀδιαίρετα), or none of them are.[140] In fact, he shows that they would be composed (or not composed) of indivisibles *in the same way*.[141] In the chapter that follows, Aristotle argues that these things are indeed continuous, and he even extends this isomorphism to the things that "follow upon" continuity: namely, he aims to show that if magnitude is *infinite*, so too is time, etc., and if time, so too magnitude—and they are even infinite *in the same way*. It is on the basis of precisely this "isomorphism thesis" that Aristotle famously claims to be able to undermine one of Zeno's arguments. In presenting his arguments, Aristotle introduces and makes significant use of the following important distinction: "for length and time, and generally everything

[140]Aristotle, *Aristotle's Physics*, 231b19-21.

[141]In "Aristotle against the Atomists," in Kretzmann, *Infinity and Continuity in Ancient and Medieval Thought*, 102, Miller calls this the "Isomorphism Thesis".

continuous, are said to be infinite in two ways: either *by division* or *at the extremities* [τοῖς ἐσχάτοις]."[142] He states that while things infinite in quantity do not admit of "being touched [ἅψασθαι]" in a finite time—i.e., some quantity that was infinite "at the extremities" could not be traversed in a finite time—those that are infinite by way of division do admit of being touched in a finite time, since time is also infinite by way of division (and in the same way). So, Aristotle concludes, it turns out that "a thing goes through the infinite in an infinite, and not in a finite, time, and touches infinitely many things in infinitely many times, and not in a finite number."[143]

It may strike the reader—at least one who has not already appreciated the distinction between the way a part composes a whole (and admits of division) and the way an (indivisible) limit limits what it limits—as very strange that immediately following such arguments, Aristotle returns to the 'now' (τὸ νῦν) and begins VI.3 with the claim that the now must be indivisible, and not just in any sense of the word, but "in its proper and primary sense [καθ' αὑτὸ καὶ πρῶτον], in which sense it inheres [ἐνυπάρχευιν] in every time."[144] Did he not just argue that time, motion, and magnitude were each continuous and divisible in the same way—and that continua were *not composed* of indivisibles? Either Aristotle has a very poor memory, or it is precisely here that we must begin to appreciate the importance invested in the distinction between what obtains for and between the parts of a whole and what obtains for and between the limits by which something is made continuous and complete. While

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[142] Aristotle, *Aristotle's Physics*, 233a27-30.

[143] Ibid., 233a31-32.

[144] Ibid., 233b37-234a1.

it is indeed the case that for and between the parts of a whole, as long as it is continuous (in the "accidental" sense of Book III), that whole must be (in one way, at least) infinitely divisible, this does not jeopardize the fact that all continua, while defined by the fact that all parts themselves have parts of the same genus, are made continuous by way of their indivisible limits, limits that among other things act to prevent anything from being infinite "at the extremities." To provide still more force to this distinction, let us follow Aristotle as he returns to consider the issue with regard to the way the now acts as a limit of past and future time:

> The now that is the extremity of both times must be one and the same, for if they are different, one could not be in succession [ἐφεζῆς] to the other, since what is composed of things having no parts is not continuous;[145] while if they are separate [χωρὶς] a time would be in between [μεταξὺ], for *every continuous thing is such that there is something of the same name* [συνώνυμον] *in between its limits* [μεταξὺ τῶν περάτων]. But if there is a time in between, it is divisible, since every time has been shown to be divisible. Therefore the now would be divisible. But if the now is divisible, something of the past would be in the future, and something of the future in the past; for that at which it is divided will mark off the time that has gone past from that which is going to be. [...] Furthermore, some of the now will be past and some of it future, and not always the same past or future. So neither will the now be the same, for a time is divided at many points. Therefore, if it is impossible for such things to belong to it, it is necessary that the now must be the same [now] that belongs to each of the two times. But if it is the same, it is clear that it is indivisible [...]. That, then, there is something indivisible in time, which we call the now, is clear [...].[146]

Again, we see that the continuity of time is in no way held to be incompatible with the indivisibility of the limit by which that time is made continuous. Nor is this indivisibility

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[145] And we know that necessarily, if continuous, then in succession. This is just one of the many instances in which Aristotle does use the fact that continuous things are also successive.

[146] Aristotle, *Aristotle's Physics*, 234a6-24, my emphasis.

of the limit by which the limited is constituted held to be incompatible with the fact that everything continuous has "something of the same name in between its limits" and thus admits in-betweens; in fact, it is what guarantees such properties. As the indivisible limit of both stretches of continuous time ('before' and 'after'), Aristotle will argue that not only is nothing "moved in the now," but "neither is it possible to be at rest [in the now]."[147] The "fusion" of the extremities of the past and future in the single limit at which these stretches meet guarantees the *strong continuity* of time. However, the time stretch that is thereby limited by the limit becomes, by virtue of having such a limit, capable of supporting 'in-betweens' and an entire metric measuring their distance from the now. In terms of such in-betweens, there can be no 'first' motion or change, and so there is divisibility.

In VI.5, Aristotle narrows in on the issue of the first into which a changing thing is changed, where "I call 'first' [πρῶτον] that which is such-and-such not on account of anything other than it,"[148] in which context we see that it is not entirely true that there can be no 'first' change. Specializing the discussion to time, Aristotle draws an important distinction between "that in which the change was first completed (for then it is true to say that it has changed)" and "that in which it first began to change."[149] As for the former, this does indeed "belong to it and exist [ὑπάρχει τε καὶ ἔστιν]" for precisely the reason that "a change admits of being completed and there is an end of change, which was shown to be indivisible on account of being a limit"; on the other hand, the alleged "beginning" of the change "does

[147]Aristotle, *Aristotle's Physics*, 234a33. The latter holds, for "we say 'at rest' that which, being of such a nature as to be moved, is not moved when, or where, or in the manner that is its nature, but since in the now nothing is of such a nature as to be moved, it is clear that neither can anything be at rest" (a34-234b1).

[148]Ibid., 235b34-35.

[149]Ibid., 236a10-12.

not exist at all, since there is no beginning of change, nor any portion of time in which it is first changed."[150] Forming something like the negative counterpart of this distinction (extended beyond the case of time) between the first in which a change is *completed* and the first in which it began to change, is the distinction between infinity "at the extremes" and infinity by division. Unlike with division, with respect to infinity at the extremities, "no change whatsoever is infinite; for every change was understood to be from something to something, those between contradictories as well as those between contraries,"[151] since for whatever is in between contradictories, the assertion or the denial is the limit, while for those between contraries, the contraries are the limits of the change—and so every change is bounded. Again, we see the pivotal dual role of the limit as at once unifying/completing by virtue of its indivisibility and also allowing for divisions to infinity.

I will conclude my discussion of the *Physics* by briefly looking at how Books VII and VIII work up to the conclusion that only the cosmos as a whole moves continuously, in the strictest sense, and that it does so only on account of the nature of the first unmoving, indivisible mover. In this manner, the relation between the first mover and the cosmos as a whole largely mirrors the structure we have been isolating throughout, whereby something is rendered continuous by being bounded and completed "internally" through the action of an indivisible and unique limit. Paralleling the discussion in the *Posterior Analytics* of whether or not there is "genus crossing," VII.4 even addresses whether or not there is transference between motions of one kind or genus and motions of another, i.e., whether all motions are

[150] Aristotle, *Aristotle's Physics*, 236a14-15; a16-17.

[151] Ibid., 241a28-30.

comparable (συμβλητὴ) with one another, as well as anticipating some issues that will be of greater concern to us in the next chapter:

> So this must be examined: what a difference of motion is. And *this argument implies that a genus is not some one thing* [τὸ γένος οὐχ ἕν τι] *but contains a manyness in it that escapes notice* [παρὰ τοῦτο λανθάνει πολλά], and some of the equivocations that there are hold a great distance from each other, while others have some similarity, and yet others are very near either generically or analogically, on which account they do not seem to be equivocal. When, then, is there a difference of species/form [τὸ εἶδος]? When the same thing is in something different, or when a different thing is in something different? And what is the dividing line? And by what do we judge that the white or the sweet is the same or different in form—is it because it manifests itself differently in something different, or because it is not the same at all?[152]

Here we again see Aristotle suggest that motions are only comparable if they are of the same sort, within the same genus. We also see another explicit acknowledgment that beyond the overriding unity to any genus, a less acknowledged aspect is that a genus "contains a manyness/plurality in it that escapes notice."

The final book of the *Physics* is dedicated to arguing that since everything that moves is moved by something; that the latter is either motionless or in motion; that when it is in motion it is always moved either by itself or by something else; and since "it is impossible that the thing moving and itself being moved should go on to infinity, for of infinitely many things none is first,"[153] there must be a first source of the things moved that does not move on account of anything else, and whose motion is continuous, one, and everlasting. At first, Aristotle observes that since this first mover appears to be moved, but not by another, it

[152]Aristotle, *Aristotle's Physics*, 248b30-249a29, my emphasis.

[153]Ibid., 256a18-20.

would appear to be "moved by itself."[154] However, he then introduces the possibility that in fact "it causes motion while being motionless," a possibility he regards as the most desirable:

> Now if that which causes motion is moved not accidentally but necessarily, and could not cause motion if it were not moved, then the mover, to the extent that it is moved, must be moved either with the same form/kind [κατὰ τὸ αὐτὸ εἶδος] of motion that it causes, or with a different kind. I mean that either the thing heating is also itself heated and the thing healing is healed and the thing causing change of place having its place changed, or else the thing healing something has its place changed and the thing causing change of place is being increased. But it is clear that this is impossible; for it is necessary to articulate a division all the way down to the indivisible [ἀτόμων] [kinds] [. . .]. Or if it is not this way, then one motion must come from the other genus [ἐξ ἄλλου γένους]; for example, the thing changing place being increased, but the thing that increases it is altered by something else, and the thing that alters it is moved with some different motion. But it is necessary to stop, since the [kinds of] motions are finite [πεπερασμέναι].[. . .] Still more unreasonable is the consequence that, since everything that is moved is moved by something that is itself moved by something else, everything that can cause motion has a corresponding ability to be moved; for it would be movable, in just the same way as if one should say that everything that can heal something is healable and everything that can build something can be built, either directly or through a number of steps. I mean, for example, if everything that can cause motion is movable by something else, but not movable with the same form/kind of motion with which it moves the next thing, but with a different kind, say, what can cause healing is capable of learning, still this, in being traced back, comes at some point to the same kind. So the one alternative is impossible and the other is like fiction; for it is absurd that what can cause change in quality is necessarily such as to be increased [i.e., a change in quantity]. Therefore it is not necessary for what is moved always to be moved by something else, and for this to be moved; and therefore it will come to a stop. And so the first thing moved will be moved by something at rest, or it will move itself.[155]

[154] Aristotle, *Aristotle's Physics*, 256a21-22.

[155] Ibid., 256b29-257a3... 257a6-9...257a15-28, translation altered.

We have built up enough scaffolding at this point to make good sense of an otherwise very challenging passage. Observe that the famous argument for the special features of the first mover argument relies, above all, on two ideas or principles: (1) the indivisibility and incomparability of distinct genera or forms of motions; and (2) what we have been calling the "internal" continuity of continua (which involves suggenicity in particular). It is also worth emphasizing a perhaps obvious point: that Aristotle is arguing for the existence of a *first* mover, so as to terminate an infinite regress, *in order to secure the continuity of motion.* That he is even making such an argument might at first sight seem strange, but it should make sense to us now that we have carefully distinguished between being continuous by way of infinite divisibility (where there is no 'first') and strong continuity defined in terms of the action of the indivisible limit that completes (and so is the 'first principle' of) that of which it is the limit.

Aristotle will go on to argue that change in place is necessarily primary of all the sorts of motion, for none of the other changes are possible "if there is not the continuous process [τῆς συνεχοῦς μὴ οὔσης] which the first mover sets in motion,"[156] and so in the strictest sense, no other motion except for change in place can be truly continuous. In spelling out this argument, and in defending the claim that "continuous motion is *possible*," Aristotle relies on two key assumptions: (i) continuity is *better* than the alternative (mere succession), and (ii) in nature we always assume the presence of the better. Ultimately, the first mover—something that is characteristically not in or a part of nature—is shown to be the sole guarantor and principle of the continuity of natural things in motion.

[156]Aristotle, *Aristotle's Physics*, 260b27-29.

Regarding the claim that only change of place is truly continuous (in the most demanding sense), it is not just any change of place that is continuous in the most demanding sense, but only motion in a circle. For a body in motion must rest at the beginning and the end of its motion; and where there is rest, there is a gap in the motion; and where there is a gap, there is not continuity. In this context of developing the strictest possible sense of a continuous motion, Aristotle returns for a final time to Zeno, specifically "whether it is necessary always to have come to the half-way point, though the half-way points are infinite and it is impossible to go through infinitely many things."[157] After repeating his earlier solution that "it is in no way absurd if someone goes through infinitely many things in an infinite time," he remarks that while this solution may be sufficient against the likes of Zeno, in fact "with regard to the underlying reality [τὸ πρᾶγμα] and the truth, it is not sufficient."[158] For, having been divided, in truth

> neither the line nor the motion will be continuous; for a continuous motion is of something continuous [ἡ γὰρ συνεχὴς κίνησις συνεχοῦς ἐστιν], and while infinitely many halves are present in what is continuous, they are present not actually but potentially [οὐκ ἐντελεχείᾳ ἀλλὰ δυνάμει]. And if [the halves] are made actual [ἂν δὲ ποιῇ ἐντελεχείᾳ], this will make the thing not continuous [οὐ ποιήσει συνεχῆ], but an intermittent motion [ἀλλὰ στήσει]. And it is clear that this very thing happens in the case of the one who counts the halves; for it is necessary for one point to be counted as two, since of one half it will be an end, and of the other a beginning, if we count not the one continuous [whole] [ἂν μὴ μίαν ἀριθμῇ τὴν συνεχῆ], but the two halves. So to the question whether it is possible to go through [διεξελθεῖν] infinitely many things [ἄπειρα] either in time or in distance, one must say that in a certain way it is, but in another way it is not. For if things [ὄντα] are infinitely many [ἄπειρα] actually, it is not possible, but it is possible if they are so potentially. For what is moved continuously has gone through infinitely

[157] Aristotle, *Aristotle's Physics*, 263a5-7.

[158] Ibid., 263a15-16; a17-19.

> many things accidentally [κατὰ συμβεβηκὸς], but not in an unqualified way [ἁπλῶς]; for it is accidental to the line to be infinitely many halves, but the thinghood [οὐσία] and the being [τὸ εἶναι] of it are different from that.[159]

Here we see one of the many instances in which Aristotle claims that the characterization of continuity in terms of infinite divisibility is *accidental*, and that there is another, unqualified, non-accidental characterization of continuity. Stronger still, "if [the divided halves] are made actual (ἂν δὲ ποιῇ ἐντελεχείᾳ), *this will make the thing not continuous* (οὐ ποιήσει συνεχῆ)." In other words, while *potentially*, a continuum is infinitely divisible, if these divisions were *actualized*, the continuity of the underlying thing in question would be destroyed. I think that this idea, typically removed from its larger context, has been incorrectly interpreted in the past. The point here, as in many other places, is that the potential infinite divisibility of a continuum is not just accidental to it, but as far as the underlying reality goes, continua are decidedly *not* infinitely divisible. The consequent alignment of continuity in the strongest sense with indivisibility, rather than infinite divisibility, has been eclipsed, over and over again, by commentators' desire to focus on the accidental feature of continua—that they are *potentially* divisible—which then serves to redirect all the deep questions about continuity and nature into purely epistemological questions. But Aristotle's alignment of continuity (in the non-accidental sense) with *indivisiblity*, and not with infinite divisibility, confirms the many other indications we have highlighted throughout this chapter of the fact that in the dual role of the limit as dividing and uniting/completing a continuum, it is the aspect of uniting/completing that is most essential. I believe that this chapter should suffice

[159] Aristotle, *Aristotle's Physics*, 263a17-b9.

to convince the reader at the very least that, for Aristotle, in the most important sense continuity was aligned with indivisibility, finiteness, and completion, and not with infinite divisibility.

Let me end this discussion with an observation. In the same way that at a "local" or more specialized level, place was defined in terms of the first unmoved limit of the surrounding body, time was defined in terms of how the indivisible now acts as the motionless (and "restless") limit of the before and after, and magnitude was defined in terms of the indivisible unit, at a "global" level, the continuity of the motion of the entire universe is itself secured by the existence of a first mover that, as a limit and first principle, is itself indivisible and motionless. At the local level, Aristotle secures the "internal" continuity of, e.g., bodies, stretches of time, and magnitudes, precisely by positing the action of a limit that simultaneously acts to "externally" discretize that body, stretch of time, magnitude, with respect to others, and to provide the boundaries of the in-between motions that are involved in changes from one extremity to its contrary. At the global level, the "internal" continuity of the moving universe as a whole is secured by the action of a limit that simultaneously acts to complete the universe "internally" and determine it as a single universe ("externally" discretizing it).

Conclusion

In the *Posterior Analytics*, at a "local" level, given demonstrations were defined in terms of the relation between the indivisible premises and that which follows from them, while at a "global" level, the coherence and completeness of demonstration as an enterprise productive of scientific knowing was secured by the kind of indivisible or "immediate" knowledge (νους) of the principles, a kind of knowing that itself acts as the limit and first principle of demonstrative knowing. In both the *Physics* and the *Posterior Analytics*, at the "internal" level of a continuum (whether a given continuous "underlying subject" or motion, or a continuous demonstration), suggenicity is the rule. In terms of the "external" discretization of these continuous object, the result is distinct kinds and their corresponding sciences, between which there is no "transit" or "transference," except in an accidental way. This common structure, found in both texts, justifies and explains the epigraph that began Chapter 1.

This makes for a vision of the universe that at once accounts for the ineliminable variety and diversity of things and for the fact that each distinct thing or general kind is individuated or determined as a 'one' by the peculiar way its parts stick together. On its own, this is already quite an accomplishment. Aristotle's account of continuity in particular is much richer than commentators have traditionally allowed, in attempting to reduce his account of continuity to what he says about the accidental sense of continuity (to which belongs infinite divisibility). As far as the continuity of nature as a whole goes: Aristotle argues for the fact that continuity in nature is even possible by appealing to the idea that "continuity is better" and "in nature, we always assume the presence of the better," an idea that would have a profound influence on many traditions to follow. While it is not until perhaps Leibniz that we see a sustained attempt to defend such a claim, this preference for strong continuity—as opposed to the weak or accidental sort we get with infinite divisibility—has exerted a profound influence over much of the tradition of physics and philosophy that has been carried out in his wake, up until today.

While one of the indirect aims of this chapter was to draw attention to some of the sophistication, richness, and lasting *aporiae* to be found in Aristotle's "true theory" of continuity, at risk of reducing some of this complexity, there is a particularly dominant feature of Aristotle's account of continuity that I would now like to highlight, as it will be of great importance of the next few chapters, in addition to being an important observation in its own right. For Aristotle, for all the nuances in his account of continuity, continuity is fundamentally about *closeness*, about relations of *nearness*. Having journeyed deeply into Aristotelian territory throughout the last two chapters, or having absorbed any of the various influential developments in the concept of continuity to have preserved this aspect of the Aristotelian account,[160] the reader might be thinking that this is almost "too obvious" to deserve mention. But it is just this feature of his account that at once continues to exert a powerful influence over many accounts of continuity that might otherwise depart from the Aristotelian theory and that would be challenged, however quietly at first, by certain thinkers and ideas to be discussed throughout the rest of this dissertation.

In the Medieval period, there were a few thinkers who would take up the challenge of developing the "continuous logic" of genera and forms at the "internal" level. It is no accident that these efforts essentially involved a re-evaluation of the relations between qualitative and quantitative changes and a closer investigation of the nature of contrariety. One such effort led to attempts to quantify what had previously been held to be qualitative by taking seriously Aristotle's notion that the contraries imposed a *measure* on the intermediate states; but, even in discussing changes to a *contradictory*, Aristotle had already made observations of the following sort:

> Nor will there be anything impossible for us in a change to a contradictory, as when something is changing from not-white and is in neither

[160]On some of these, see the concluding chapter of this dissertation, where the provisional "classification" of the concept of continuity is presented.

> condition, on the ground that it is therefore neither white nor not-white. For it is not because something is not as a whole in either condition that it would not be called white or not-white, since we call something white or not-white not through its being wholly so but because most or the most important of its parts are so, and not to be in a certain condition is not the same thing as not to be in this condition wholly. And it is the same with being and not-being and the rest of the changes to a contradictory, since the thing will necessarily be in one or the other of the opposites, but all the time is in neither wholly.[161]

While this sort of remark is entirely consistent with the overall picture we have painted of Aristotle's approach to continuous change, it is arguable that Aristotle did not have the "logical resources" to actually take such intermediate conditions as seriously as possible, or to consider the full implications for ontology and metaphysics of taking them seriously. Certain Medieval thinkers would invent more refined logical tools in order to begin to more fully incorporate these sorts of intermediary stages not just into what was held to be knowable, but into a vision of nature that was decidedly more nuanced. One major issue lingering from Aristotle, however, is that Aristotle occasionally made remarks to the effect that, in a more demanding sense, and unlike change in quantity and change in place, "only in motion with respect to quality [κατὰ τὸ ποιὸν] is it possible for there to be anything indivisible in its own right [καθ᾽ αὑτὸ]."[162] On account of remarks such as these, many of the efforts to make "logical room" for the sorts of continuous objects and changes that allowed, "internally," for intermediate states, were focused through debates concerning whether or not the degrees of a quality in particular, i.e., the degree to which one extreme of a contrary was possessed, were indivisible. This concern was further related to how, given that we are dealing with

[161] Aristotle, *Aristotle's Physics*, 240a20-32.

[162] Ibid., 236b16-19.

contraries, the latter could even be held to inhere in a self-same “underlying subject” and how there could be an intension or remission of a quality without this destroying the identity and invariance held to belong to forms.

With Duns Scotus in particular, we see an especially powerful attempt to re-focus attention onto the sort of continuous motion that was mostly ignored by the tradition that followed in the immediate wake of Aristotelian physics, namely qualitative motions. Closer attention to the determination of qualitative changes in relation to the nature of magnitude involves Scotus in the debates concerning the intension and remission of forms, where things like age, ripeness, loudness, color, and charity are understood to be qualitative or “formal” alterations that necessarily involve intermediary states, internal variability, and changes in degree. Reconciling this with the traditional view of the invariance and indivisibility of forms would lead to significant revisions in the notion that variability between the extremes of contraries can be made to inhere or take place in a single, self-same entity—ultimately leading to efforts to take more seriously the idea that certain forms or qualities support an interval of variability. As we will see, the quantification of quality, in Scotus’s hands, will involve a profound transformation of the concept of quantity. In this connection, his “modal distinction”—meant to capture the idea that some natures come in a range of degrees and that this range is inseparable from *what* they are and is “intrinsic” to them (for precisely the reason that they are differences that belong within the boundaries laid out by a given “formality” or quality)—pushes to the limit Aristotle’s suggestions that qualitative changes in particular support a range of “in betweens” and so allow for continuous variation without thereby destroying the identity or oneness of the underlying thing changing. In this respect, one of Scotus’s more ambitious aims concerns his efforts to demonstrate an *identity* between ranks of essences and quantitative degrees of power/perfection. The connection between this endeavor and the transformations in the concept of continuity as this relates to *generality*, as well as some of the difficulties in how this "fusion of magnitude of perfection/power with the essence" can give rise to a more refined notion of *measure*, is made even more explicit in the work of Oresme, to which the latter half of the subsequent chapter is devoted, and where we will discover perhaps the earliest deliberate, if somewhat subtle, challenge to the Aristotelian alignment of continuity and closeness.

Chapter 3

Scotus and Oresme

Introduction

In some cases, we either have something or we do not, something is or it is not. A cup of coffee is on my desk or it is not; an earthquake happens or it does not; a rock falls or it does not. However, one does not need a degree in philosophy—or anything for that matter—to appreciate the fact that many (if not most) aspects of reality are not like this. Minerals each exhibit a certain "hardness," but some are "harder" than others. Two avocados may both have a certain "ripeness," but one may be *more ripe* than the other. Sounds have a certain loudness, but one sound may be *louder* than another. Certain colors may each have "redness," while one is *more red* than another. One cup of coffee may have a certain "heat" or "hotness" that is greater than another cup or than itself at a later time. Earthquakes have a certain intensity, but some may be *more intense* than others or after a certain time. The degree to which we have "charity" or are charitable seems to change over time, just as some people seem to be *more charitable* than others. We do not begin to understand these sorts of aspects of reality if we are only prepared to consider whether or not they are there. It seems to be an *essential* component of these things not only *that* they come in degrees, but just what degree they come in, for this seems to be inseparable from the thing. These are not differences that we can "will away" or attribute to the mind or relegate to some "accidental" status, try as we may. You can scratch gypsum with a knife or even your fingernail, but you will not scratch diamond with a knife; you cannot drink a scalding cup of coffee without burning yourself; try convincing a person whose home was destroyed by an earthquake of great magnitude that "an earthquake is an earthquake," and that you went through the same thing when you once experienced a slight tremor; or try convincing yourself that your extremely inconsistently charitable, or less charitable friend, is just as good a companion as your highly charitable friend, because "charity is charity." The differences in degree of such things, we almost want to say, make *all the difference*. And yet. . .

In another sense, it is true that, well, "charity is charity" and an "earthquake is an earthquake" and "red is red"—after all, for all their differences, one shade of red seems to have a lot more in common with another shade of red than with an earthquake, and an earthquake of magnitude 3 seems to have a lot more in common with an earthquake of magnitude 9 than with some shade of red. These sorts of observations almost seem too obvious to mention. Yet our inveterate desire to reduce things to "either being there or not" has made it very difficult to understand this sort of thing, i.e., what it is like to be a thing that can be *more or less* and still be *the same thing*. What is it like for a reality to have a unity and definite boundaries while admitting many (infinite?) degrees of variation? Moreover, even if we want to somehow account for the important *differences* between a magnitude 3 earthquake and a magnitude 9 earthquake or between one shade of red and another or between a highly charitable person and a mildly charitable person, we must also account for the very *real commonness* that one earthquake has with another, less intense earthquake, something it does not share with a color of any intensity. This commonness among such things that admit of variation in degree, moreover, does not seem to be a matter of different instantiations of a single *universal*. Among other reasons for not thinking about it in this way (some of which will be seen in this chapter), to think that would be to ensure that any differences in degree were *extrinsic* to the thing and somehow became attached to its nature on account of the confluence of a number of other contingent circumstances. Yet it seems more accurate to say that the fact that certain natures (such as those mentioned above) come in degrees is an integral component of not just *how* they are, but *what* they are. To be a thing that can be more or less is to be a special *sort* of thing, to enable a certain type of criterion of comparison. Even as the differences in degree can be easily distinguished, there is a kind of unity among the different degrees of such things, without that unity being provided by the standard universal or by the action of a mind.

Even if we allow that the differing degrees of such things make for important and ineliminable *differences*, it also seems true that sound "uncontracted" to any one degree of intensity is still a definite and definable reality, just as what an earthquake *is* is something that can be precisely defined without having to specify any one degree of magnitude earthquake, and just as there is a sense in which "charity is charity" regardless of degree, as it seems to be something that (at least in principle) can be given a precise definition without one needing to specify its degree (even if it remains true that we will only ever "meet" charity as already "contracted" in a person or in ourselves to some particular degree of intensity). In short: we recognize that things like charity are the sort of thing that can be had to a greater or lesser degree; but we are also tempted by the idea that "charitability" or "charity," like "ripeness" or "redness," is the sort of thing that is *invariant*—in the sense that we might think that while we can become more or less charitable, while fruits can become more or less ripe, and while bodies can become more or less red, whatever charity or ripeness or redness *is*, it will continue to be this regardless of the many degrees through which it passes or in which it is exemplified. Such thoughts might tempt us almost to say that "charity itself" does not change, that the changes in degree charity admits are somehow "extrinsic" to what charity "really is." But is this actually a coherent way of thinking about things? What, really, does it say about things like "charity" to say that it can be *more or less*? Surely there is a difference between the acquisition of charity (where, prior to this, we did not have the capacity to be charitable) and the process of coming to have charity "more fully"? But how can something come to be had "more fully" without thereby changing, without becoming something else? Moreover, in becoming more charitable, for instance, we appear to become *better*—not just as people, but as *charitable*. But how can something become *more*—and in becoming more become better—without becoming *another*? What if, in fact, in becoming more, certain things become "more themselves"? What kind of a vision of the world would we be committing ourselves to in saying such things?

These sorts of questions were of vital concern to Medieval philosophers, who recognized that some realities, like the sorts of things mentioned in the previous paragraph, came in degrees, admitted a 'more and less', but were also *forms*, and as such, ought also to be *invariant*. Thus, if one holds that it is the "form itself" that increases or decreases, one would seem to have to undermine this alleged invariance of the forms; yet if one tries to make these "increases or decreases" somehow accidental or extrinsic to the form, one would seem to rob oneself of the ability to treat such differences in degree as in any way important to what each form *is*. The implications of how one answers these questions and the theory one provides to account for such things, run very deep and very far, touching on all kinds of other issues. For instance: what happens to how we must think of *quality* and *quantity* when qualities and forms are *quantified*? How can there be unities (not of the "universal" type) that embrace different degrees, that allow for a characteristic *spectrum* of realization? How can a form (traditionally regarded as *indivisible*) be divisible into degrees (and not just "divisible in thought," for that would be to make such differences in degree a *mental* matter)? How can something indivisible be an interval of variation? How can some things, in becoming *more* of what they are—and not by adding on extra "copies" of itself—even become *more perfect* in relation to other things?

The above sorts of questions are, at bottom, questions about the conceptualization of continuity. What came to be called the "latitude" of a quality referred to an interval of variation, a range that, in admitting of differences in degree, formed something of a continuum. While Aristotle regarded alteration (qualitative change) as continuous, it is not immediately clear how a quality or form can change degree, continuously, without the quality or form itself changing what it is—thereby inviting new theories about how certain natures (like certain forms and qualities) could support an interval of variation and remain themselves, i.e., that certain natures admitted of degrees and that this was inseparable from *what they are*. Further questions would naturally arise as to how the degrees of the sorts of forms and qualities with a "latitude" composed with one another, whether or not such degrees were to be regarded as themselves indivisible, and so on. The Aristotelian account of both continuity in general and alteration in particular had set out a number of such problems for which the concept of intensity or *intensive magnitudes*—emerging out of the fact that certain qualities could become more or less of themselves—was to become, in the hands of certain Medieval philosophers, the principal solution.

Duns Scotus took special interest in the main question concerning the increase and decrease of qualities, in the course of which he advanced a number of subtle and profoundly transformative ideas. Perhaps the most transformative of these was his notion of a "transferred" sense of quantity, on the basis of which he would develop his formal and modal distinctions, and the concept of intensive modes as "inseparable from the nature of a thing." As the continuous variations in degree modified a certain "formality" and so were held to be "intrinsic" to it and unified by the *ratio* of that form, Scotus's ideas on these matters would further pave the way for a more nuanced understanding the phenomenon of generality (via his "common natures"). These ideas would be pushed to their extreme by Nicolas Oresme, a half-century later, in his radical attempt to geometrize the measure and comparison of intensities with his "figuration of qualities." According to Oresme, since "every measurable thing except numbers is imagined in the manner of continuous quantity," it follows that "for the mensuration of such a thing, it is necessary that points, lines, and surfaces, or their properties, be imagined," for it is with such geometrical things that one first finds measure and ratios.[1] Thus, Oresme wagers, every "intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible things, e.g., a quality," and "whatever ratio is found to exist between intensity and intensity, in relating intensities of the same kind, a similar ratio is found to exist between line and line, and vice versa."[2] In setting up this correspondence between intensities and lines, and between ratios of intensities and ratios of lines, Oresme holds that

> just as one line is commensurable to another line and incommensurable to still another, so similarly in regard to intensities certain ones are mutually commensurable and others incommensurable in any way because of their [property of] continuity. Therefore, the measure of intensities can be fittingly imagined as the measure of lines, since an intensity

[1] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, I.i.

[2] Ibid.

> could be imagined as being infinitely decreased or infinitely increased in the same way as a line.[3]

Oresme built on top of this subtle "figuration of qualities"—grounded in the "fitting" correspondence between intensities and continuous quantities "figured" by geometrical objects like lines and surfaces—the bold idea that the "configurations" of multiple intensive qualities and the relations between their respective ratios of intensity—more precisely, the fact that certain "configurations of qualities are mutually conformable or fit together better while others do not fit together well"[4]—were the principal *causes* of the "fitting accord [*convenientia*]" or "natural friendship" and "discord [*disconvenientia*]" or "natural hostility" that existed between different entities (in particular between different species) throughout nature. In addition to his profound realization of certain of Scotus's ideas concerning the intension and remission of qualities, Oresme extends these ideas even further, initiating a still more radical transformation in the concept of *continuity*. For Oresme,

> it is certain that some ratios seem naturally to accord [*convenire*] better with each other, just as a rational ratio accords better with a rational ratio than it does with an irrational or surd, and a harmonic ratio better with another harmonic than with an non-harmonic [*enarmonica*] ratio. So it is also, in regard to figures, that certain of them are more conformable to, and consonant with [*magis conformes et magis consone*], each other than to others. Thus when one [of such conformable figures] is inscribed in, or circumscribed about, the other, or compared to it in some other way, it is related to its comformable figure more beautifully than it is to some other figure. As for example, perhaps a square is more beautifully related to a circle or an octagon than it is to a pentagon. The same thing is true regarding the aforesaid configura-

[3]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, I.i.

[4]Ibid., 243.

> tions of qualities: some are mutually conformable or fit together better while others do not fit together well.[5]

Taking his "figuration" of intense qualities with continuous quantities (represented by lines, curves, and surfaces) and building a theory of "agreement" and "disagreement" between natures based on comparison of the ratios of such intensities and their "figurations," in Oresme's hands the measure of the continuous is quietly released from its hallowed alliance with *closeness* and its accompanying distance metrics—and so its inevitable bondage to *extrinsic* specifications (in terms of proximity or similarity)—so as to allow continuity to take on a sense of *structural conformity or accord* between different ratios of intensive qualities and the relations of their configurations. In this way, continuity becomes less a matter of closeness and more one of morphological affinities or conformity.

> Just as we see in music that it is not by the degree of closeness of one sound to another that it is more harmonious with it but rather that a fitting ratio is required, *so here closeness is not the attendant measure but rather a fitting and natural conformity.*[6]

In insisting on the "fitting accord [*convenientia*]" as supplying a type of unity that is "not that of closeness, but of conformity [*non propinquitatis sed conformitatis*]"—and not as one of "similarity" either[7]—Oresme frees the concept of continuity from its ancient connection to *closeness* in both the sense of proximity and superficial similarities. Oresme's transformation of the concept of continuity comes at two levels. First, his powerful ideas for the representation of intensities by continuous geometrical "figurations." Second, specifically in showing

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[5]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 241-3.

[6]Ibid., 243.

[7]Ibid., 245.

how there are certain distinct generic *types* according to which intensities vary, Oresme can begin to compare the ways in which different subjects change intensively, and to compare them the way one would compare continuous quantities in geometry. This leads to his theory of concord or consonance of certain natures through the *conformity* of their configurations and their respective ratios of intensities (themselves compared as one compares continuous quantities). This amounts to a picture of nature as shot through with morphological continuities and discontinuities, grounded ultimately in how well certain ways of changing in intensity compose with other ways, i.e., in whether one of the "figures" (corresponding to a certain way of changing in intensity) can be "inscribed in, or circumscribed about, the other, or compared to it in some other way [and related to it in a more consonant way]." In these two ways, continuity can now be treated as (1) an *intrinsic* property (of a single given quality or form changing in intensity and supporting a range of degrees); and, in passing to the *relations* between the distinct types of intensive changes, as (2) a matter of morphological "conformity." It is above all in this second respect that Oresme detaches or "frees" continuity from *closeness* in the name of conformity between the ratios of intrinsic determinations and their figurations, leading to a notion of *morphological (or structural) continuity.* This "freeing" of continuity from closeness, lifting it towards a more structural account in terms of relations of conformity between the various types or "figures" of intensive changes, is perhaps the single greatest and furthest-reaching transformation of the concept of continuity.[8]

[8]Incidentally, I hope to show in the section devoted to Oresme that other issues such as whether Oresme anticipated Cartesian coordinates and the notion of functional dependence, and whether he anticipated any number of other "modern" scientific advances, pale in comparison to the profound and lasting impact of his transformations to the concept of continuity.

Brief Background

In the medieval period, a distinction was often made—one usually traced back to Augustine—between *quantitas molis* (quantity of bulk or stuff) and *quantitas virtutis sive perfectionis* (quantity of power or perfection). Roughly, the former was identified with extensive quantity, while the latter sort of quantity involved those who dealt with it in efforts (i) to quantify what had been held to be qualitative (and to reconcile this approach with the problems it made for traditional but often equivocal axioms regarding, for instance, the ontological status of forms and accidents); and (ii) to understand how the degrees of a quality were ordered in terms of perfection and how power or perfection could be had to an infinite degree (and thereby ultimately to solve the problem of conceptualizing God's infinity, for instance, as distinct from indefinite spatial or extensive magnitude). The first of these efforts (quantifying qualities or accidental forms)—an effort sometimes traced back to Lombard's *Sentences*, specifically its discussions of charity (*caritas*)—came to concern a collection of problems in natural philosophy grouped under the name of "latitude of forms," or the intension and remission of forms, theories largely concerning changes in degree or the intensity of qualities. Aristotle had considered the possibility that certain virtues or a condition like health, or certain qualities more generally, could admit of a 'more and less', though this tradition believed him to have been insufficiently clear regarding whether it is the self-same form or quality *itself* that increases or decreases in intensity, or if it is merely a question of a subject coming to be qualified by a succession of *different forms*. At least by the fourteenth-century, the debates over the intension and remission of forms had become of central concern to many. Such debates were also often traced back to Aristotle's distinctions of motion from the

Physics—motion in place (locomotion), motion in quantity (augmentation and diminution), and motion in quality (or alteration).[9] Aristotle had taken motion in quality (alteration) to be a change that occurs between two extremes or contraries, between two intensities: "change within the same form to more or less is alteration, for it is motion either from a contrary or to a contrary, either simply or in some particular way."[10] Accordingly, the only motion that could properly be described as supporting an intension or remission is that of alteration, occurring as it does between two extremes; however, some began to think of even these sorts of changes on the model of local motion.[11]

There are a few main issues of note here. First, the traditional Aristotelian understanding of *substantial* forms took such forms to be both *simple* and *invariable*. In considering qualities that admit of a more and less, then, one strategy was to insist on a sharp distinction between those essential features characteristic of substantial forms, like rationality, which could not support such intension or remission, and those qualities and habits that admit a more or less, and which were accordingly regarded as *accidents*. Authors such as Aquinas would also consider how intension and remission differed for different sorts of qualities: corporal qualities such as health or local motion, divisible in themselves, were held to capable of undergoing an intension and remission in their forms, for they were held

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[9]The fourth kind of motion, generation or destruction, was often treated in this period as the limit of alteration; thus many discussions often mention only three kinds of motion.

[10]Aristotle, *Aristotle's Physics*, 226b2-3.

[11]Already by the time of Oresme, questions involving intension and remission became closely connected to questions concerning the classification and treatment of (local) motion and velocity. While historically closely related, there was nevertheless an important distinction between inquiries into intensive changes with respect to qualities held to be "permanent" and those dealing with special cases of local motion (specifically involving velocity).

to be increased by the addition of parts; on the other hand, qualities such as color or heat, held to be indivisible qualities, could undergo intension and remission only with reference to a subject, in relation to which they changed by "the varying participation of a subject in a given, unchanged quality."[12] This enabled the "participation" theory, which found another advocate in Romanus, according to which intension and remission was seen as the product of the extent to which the subject participates in the qualitative form.

But others were not satisfied with the attempt to reduce this question to one of inherence in a subject. Authors like Henry of Ghent regarded intensive changes as changes within the specific form itself, which was accordingly not held to be simple, but a divisible extension. Another popular position of the time was that of Godfrey of Fontaines, who maintained, in contrast to both Aquinas and Henry of Ghent, that all qualities or forms are indivisible and invariable and thus cannot admit of more or less in themselves. This led him, and others, to hold that each individual form is numerically distinct and so, when there is a change, it is in fact a "succession" of numerically distinct forms "replacing" one another in the subject—thus giving the name "succession of forms" to this theory.

Perhaps the most important underlying issue here is that for Aristotle, when the place, quantity, or quality changes in a single motion, the place, quantity, or quality acquired or lost in the change must be continuous, and so too must the motion itself be continuous, an assumption shared by nearly all natural philosophers by the fourteenth-century.[13] However, at least for some scholastics—most notably those who adhered to the succession-of-forms

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[12] Aquinas, *The Summa Theologica of St. Thomas Aquinas*, I-II, q.52, art.1,2.

[13] See Edith Sylla's essay in Kretzmann, *Infinity and Continuity in Ancient and Medieval Thought*, 231.

theory of alteration—the model of local motion, where the moved body occupies a different place at each instant, strongly suggested that perhaps in cases of alteration a new and indivisible degree of form or quality was generated at each instant. According to this approach, there would have to be an entirely new accidental form with any change in intensity, such that "each momentary state does not proceed from the previous one but only follows and displaces it."[14] On this view, qualities acquired or lost in alteration could not be held to be continua containing parts that could be added or subtracted gradually. Every qualitative form is held to be indivisible with respect to degree, each old degree destroyed in alteration and replaced by the new one. Those who defended this view were thus often concerned with defending this theory from being subsumed by atomism.[15] They faced the difficult task of simultaneously explaining alteration in terms of indivisible degrees of a quality and holding on to the Aristotelian axiom of continuity of motion. As was to be expected, this would present great difficulties. For, as long as one stuck closely to Aristotle's definitions of continuity, it would be futile to attempt to *compose* out of indivisibles anything resembling continua.

According to another theory—the "addition theory"—the intension or remission of forms or qualities was held to occur by the addition or loss of parts or degrees of the quality. Accidental forms of contrary qualities were thought to be capable of existing simultaneously and with varying intensity in their subject (at least under the assumption that a given sum of degrees of the intensity of both qualities remains constant), something that also led to a more subtle way around some of the cruder traditional interpretations of the extent of

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[14]Maier, *Zwei Grundprobleme Der Scholastischen Naturphilosophie*.

[15]See Kretzmann, *Infinity and Continuity in Ancient and Medieval Thought*, Chapter VIII, "Infinite Indivisibles and Continuity in Fourtheenth-Century Theories of Atomism".

Aristotle's commitment to the principle of non-contradiction, allowing for the simultaneous "mixing" of contraries. Typically held to be the main proponent of the addition theory, Scotus accordingly did not hold degrees of a quality to be indivisible, nor did he think that the question of intension and remission of forms could be reduced to a matter of inherence in a subject—and in this way, he seems to have been especially concerned with securing the continuity of qualities and forms themselves, or "internally," as well as the continuity of alteration (changes in the intensity of a quality). Oresme, for his part, was also keenly aware of the close connection between providing a sound theory of the intension of a quality, such as heat or charity, and the issue of resolving how contrary qualities can inhere in something simultaneously. Oresme has perhaps one of the most sophisticated ways of addressing the issues surrounding intension and remission.[16]

In connection with the question of how contraries could inhere in something simultaneously, recall our discussion of Aristotle's notion that within a given genus the maximally extreme contraries provide the measure or standard of comparison against which "in between" states could be measured and from which changes proceeded. Recall also how within each given genus, spanned by the interval forms from the extreme contraries, there must be in-betweens—where "all in-between things are in the same genus as the things they are between"—and so a genus can be understood to be something of a continuum (and, indeed, to support continuous variation). In the medieval period, there were some, like Scotus, who

[16]As far as the previous options go, Oresme seems to have held the addition theory to be the most plausible option; however, in the end, he seems to regard it as the second-best theory, that is, second to his own theory—called the "succession-of-condiciones" by Kirschner. In the section of this chapter dedicated to Oresme, I do not focus on this particular aspect of things, choosing instead to develop a number of other aspects of his broader configuration theory. For more on the medieval debates surrounding the degrees of a quality and the various options, the reader can consult Maier, *Zwei Grundprobleme Der Scholastischen Naturphilosophie*, 1-109.

appear to have taken up the challenge of treating certain realities that were in some sense "general" as continua. The "formalities" and qualities considered by Scotus emerge as intervals of variation, as admitting of degrees, and also as providing a kind of unity to these degrees. Scotus's notion of certain formalities as supporting "intrinsic modes," together with his concept of "common natures," add yet another aspect to this story connecting continuity and generality.

In what follows, we will build towards Oresme's radical transformation of the concept of continuity, in the course of which we will address in more detail some of the underlying issues discussed in these introductory pages. First, though, we will look more closely at how Scotus addressed the question of the intension and remission of forms and how this informed, and was informed by, his treatment of related questions. The discussion of Scotus is broken up into three main sections. First, I briefly discuss his treatment of some of the different theories accounting for the intension and remission of forms/qualities. Aside from the intrinsic interest of some of this material, this section also motivates the introduction of the formal-modal distinctions and some of the more general features of the Scotistic "world"—such as the notion of intrinsic modes and the transformed sense of quantity—to which the second section is devoted. The third section provides a brief look at his notion of "common natures."

Duns Scotus

The 4 Options According to Scotus

In the course of Scotus's discussions of the degrees of a quality and their changes—discussions that, typically, follow Lombard in setting out from questions surrounding "the manner of increase in charity"[17]—he takes up three possible theories, provides reasons for rejecting each, and then proposes his own, fourth theory. Assuming the theories are successful at accounting for how a certain quality can even coherently admit of degrees, each theory is concerned, perhaps above all, with how the 'lesser' degrees are related to the 'higher' degrees. The three theories he addresses and argues against can be summarized as follows:

1. Each different degree of a quality is explained as a different *species* of that quality. A change from one degree to another is just the complete "corruption" of the first degree and complete "generation" of the other degree (so that no part of the first degree is a part of the other, i.e., different degrees have no real parts in common).[18] (Godfrey of Fontaines and Walter Burley)
2. Each different degree of a quality is explained as a different degree of *participation* in that quality by a substance. A change from one degree to another is due to a substance participating in the quality to a greater or lesser degree. (Aquinas)
3. Each different degree of a quality is explained as a different degree of the quality's *actualization*. A change from one degree to another is due to an increase or decrease in that quality's actualization. As discussed by Cross,[19] this second claim was, in turn, traditionally developed in two distinct ways:
    a) A change from one degree to another is due to an increase or decrease in the quality's actualization, in such a way that the 'higher' degree contains

[17]For instance, Scotus, "Ordinatio," 1.17.2.1.

[18]"Here it is said that nothing of the preexisting charity remains the same in number in the increased charity, but rather the whole [*totum*] of what existed before is corrupted and another individual more perfect than it is generated [*aliud individuum perfectius illo generatur*]" (ibid., 1.17.2.1, n.198).

[19]Cross, *The Physics of Duns Scotus*, 172.

actual parts that existed only potentially in the 'lower' degree. (Henry of Ghent)

b) A change from one degree to another is due to an increase or decrease in the quality's actualization, in such a way that the whole 'higher' degree existed only potentially in the 'lower' one. (Unidentified Opponent)

After discussing these theories in more detail, and looking briefly at some of Scotus's responses to them, I will discuss Scotus's own theory, in the course of which many key Scotistic concepts will emerge and come to play a rather pivotal role. It will turn out that the most important of Scotus's responses, both in terms of where we are headed and in terms of the development of his own theory, comes in his treatment of the first of these three theories. I will thus very briefly discuss the second two theories before dedicating a more substantial discussion to the first theory, a discussion that will in turn take us more deeply into a number of more general Scotistic concerns.

Option 3: Henry of Ghent's Theory of Actualization

This theory has it that a difference in degree of a quality is due to the degree of that quality's "actualization," and certain proponents want us to believe that this entails that any lesser degree will contain ("potentially") all greater degrees. An increase from one degree to another, in being explained by an increase in the actualization of that quality, was more precisely held to stem from the actualization of already existing parts that had previously existed potentially in a lower degree. How this theory conceives of this process of "actualization" and how Scotus argues against its cogency is not something that is vital for the purposes of this chapter, and so I refer the interested reader elsewhere.[20] To the extent that there are redeemable features of this theory, they will resurface in Scotus's own preferred "addition theory," which I will treat separately.

Option 2: Thomas Aquinas's Participation Theory

One strategy of this theory, according to which changes in the degree of a quality are explained in terms of changes in the degree to which a subject "participates" in that quality, is to argue, as did Godfrey as well as Aquinas, that this degree of participation is to be further explained by the degree to which a subject is "disposed" to receive an accidental quality or by a removal of the opposition "indisposition."[21] Scotus's first rejection of this "disposition" account proceeds thus:

> Against this position I argue first as follows: contraries when extreme cannot coexist [*sunt incompossibilia*] in the same thing, but they can [coexist] when in mild degrees [*in gradibus remissis*]. But this is only because there is something in the intense contrary [*in contrario intenso*] that is not in the mild one [*non est in remisso*]; for if the whole reality [*tota realitas*] that is in the mild degree is in the intense one, then there is no repugnance between the mild and the intense degree. – But this incompossibility is not a relation to the subject nor is it from any relation to the subject [*nec ex habitudine aliqua ad subiectum*]; for the incompossibility of forms in themselves [*in se*] is prior to the incompossibility of forms in some third thing, as in the subject that receives them (for it is because they are incompossible in themselves that they cannot be received in the same subject—and not contrariwise). Therefore, that which is the reason for their incompossibility in intensity [*in intensis*] is something in them that is positive in itself and not just in relation to a subject.[22]

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[20]The interested reader can consult Cross, *The Physics of Duns Scotus*, 183-6.

[21]See Scotus, "Ordinatio," 1.17.2.2, n.241.

[22]Ibid., 1.17.2.2, n.242.

Scotus goes further with this re-orientation of the problem away from the subject and back to the intensities of the forms themselves, emphasizing that "it is because the form is such that what has it [i.e., a subject] is said to be such in accord with it."[23] One of the issues here is that if the form is supposed to be indivisible, as traditionally it was held to be, then this raises a further question, for

> if the form is indivisible it gives indivisible being to the subject and does so indivisibly; for the subject is not of such a sort in form save because the form is of such a sort; therefore if the form is of such a sort in itself, there is no 'moreness' [*maioritas*] of perfection to it, and the subject that accords with it will not be said to be more such in accord with it.[24]

The idea here, as further explained in the interpolation, is that "if the form is indivisible in itself, it is impossible that the subject could be more or less in accord with it; for it is contradictory that a 'form in itself' be indivisible and yet that the subject be divisible in accord with it." Apart from the obvious fact that, for Scotus, these questions must evidently be resolved for the "forms in themselves" rather than at the level of their relation to a subject, one of the main problems with the "disposition" account is that even if we allow that it makes sense to turn our attention to the disposition of a subject, the degrees of a disposition must itself be explained. But it would seem that it can only be explained in either of two ways: (i) the disposition that a subject has for a quality is itself a further quality of that subject; or (ii) the degree of disposition of a subject for some quality is the result of a degree of a more fundamental quality of the subject (e.g., degrees of heat could

[23]Scotus, "Ordinatio," 1.17.2.2, interpolation to n.244.

[24]Ibid., 1.17.2.2, n.244.

be explained by invoking degrees of dryness, since dryness disposes a subject to heat).[25] A similar problem faces those who look to explain things by appeal to an "indisposing" form, for the latter admits of degrees, and these cannot be explained by supposing another indisposing form. Related to these two accounts of "disposition" or "indisposition" is a third account which appeals to "rootedness" or "rooting down of form in the subject [*radicatione formae in subiecto*]."[26] Scotus rejects this third option as well, on account of the fact that in "explaining" the different degrees, it itself needs an explanation, similar to the cases of disposition and indisposition.[27]

Option 1: The Godfrey and Burley Theory of Corruption-Generation

The idea that

> (A) a change from one degree to another is the "complete corruption" of the first degree and "complete generation" of the other degree—where, in the case of *increase*, the latter generated "individual" is "more perfect" than the former—such that no part of the first degree is a part of the other

seems to have had defenders in Godfrey and Burley. Scotus's approach to this theory is to show that the account presented in (A) really rests on an implicit commitment to the further idea that

> (B) each different degree of a quality is explained as a different *species* of that quality.

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[25]The example is taken from Cross, *The Physics of Duns Scotus*, 182.

[26]Scotus, "Ordinatio," 1.17.2.2, n. 247.

[27]The reader who desires to learn more about this option in general or any of the variants, can either consult the relevant texts by Scotus, or Cross, *The Physics of Duns Scotus*.

Scotus then argues against this option as a whole by showing the inability of (B) to secure the *continuity* of changes (specifically, of alterations). In the course of treating this option, he proposes at least six counter-arguments; but arguably of greater impact (in terms of the genesis of his own theory) is his preliminary discussion in which he lays out some of the reasons one might have for taking (A) to be true in the first place, directing us to how Aristotle had upheld in *Physics* V (227a7-10) that the "terms of motion" (*termini motus*) are "incompossible." The opponent then uses this passage as a basis for treating the degree that precedes and that which follows in the case of an alteration as such "terms" of the change; and so, the reasoning goes, the terms of this motion will be incompossible terms, which cannot be "together at the same time"; therefore, it is impossible that one of the end terms (the "terminus 'ad quem'") includes the other (the "termino 'a quo'") as a part; and thus, such a motion does not include anything the same in number, and no part of the one degree is a part of the other.[28] But why should we believe this? Scotus suggests two further reasons the opponent might have for finding "confimation" of (A): the first comes from comparison with what occurs in the case of species, and the second from comparison with what occurs in the case of substantial forms. More specifically, the assumption is made that how the "hierarchy of perfection" works in either of these cases is the same as how it works for the degrees of a quality.

1. *Argument I from Simplicity: Comparison with Order of Perfection Among Species*:

   In the case of species, the "positing of the more and less" is derived from the

[28]Scotus, "Ordinatio," 1.17.2.1, n.199.

"essential ordering of species," which exhibits a "hierarchy of perfection."[29] Presumably on the grounds that the degrees of a quality also display something like a hierarchy of perfection—one intensity of a form or habit such as charity can be "more perfect" than another degree of that form or habit—it may be conjectured that there ought to be a similarity (or even isomorphism) between the case of species and the case of individuals belonging to the same species (and so "individual degrees of a quality," on the further assumption that the degrees of a quality are like the "individuals" of a species), i.e., that "the more and less in the same species are related in the same way that the more and less in diverse species are proportionally related."[30] However, this line of thinking continues, a more perfect ("greater") species is simply a different nature from a more imperfect ("lesser") species, and as such, nothing that is "of the inferior nature" remains, as the same in number, in the superior nature, for the superior nature is in fact "in itself simpler than the inferior, because in the case of forms the superior is more perfect and more actual and simpler."[31] In other words, a more perfect species will be less composite than less perfect species, and it is impossible for the less composite to include the more composite; thus, it is impossible that a more perfect species "include" a less perfect species. Therefore, assuming the same sort of thing occurs for individuals of the same species (and so too for individual degrees of a quality

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[29]Scotus, "Ordinatio," 1.17.2.1, n.200.

[30]Ibid., 1.17.2.1, interpolation n.200.

[31]Ibid., 1.17.2.1, n.200.

or form), nothing of the "less perfect" remains the same in number in the "more perfect," for otherwise the more perfect would have to be more composite than the less perfect. In short: by comparison with the case of the ordering of diverse species, it seems that within the same species the simpler form will be the more perfect and so will not possess "a preceding degree or form added to it"—so, in an increase, no part of the first degree is a part of the other, i.e., different degrees have no real parts in common.

The overall strategy of one who reasons thus seems to be to implicitly appeal to the Aristotelian idea that the indivisibility of species is modeled on the indivisibility of number, for the latter is indivisible in so far as addition or subtraction always produce a different number altogether. Similarly, the addition or subtraction of characteristic features will yield a different species altogether. So, if each degree of a quality is regarded as a species of its own, it will be indivisible in this way, and so any change from one indivisible degree to another indivisible degree can only be a complete corruption of the first degree and a complete generation of the second, where no part of the former is a part of the latter (for they are in fact effectively indivisible, or without parts). Thus, according to this reasoning, (B) entails (A).

2. *Argument II from Simplicity: Comparison with 'More or Less' in Substantial Forms*: The 'more and less' in accidental forms are related in the same manner to how 'more and less' are related in substantial forms (assuming there even is a 'more or less' in substances); yet "according to everyone who posits a more or less in substantial form," the more perfect substance, even within the same

> species, will be *simpler* than others, and so does not “possess in itself the ‘less’ [perfect] as some part of itself, but by being as simple as or more simple than the ‘less’ [perfect].”[32] It should then be the same, this reasoning goes, in the case of accidental form, thus leading again to (A).

Both of these arguments depend on the same fundamental assumption regarding the alignment of perfection with simplicity: the more perfect a species (or substance), in comparison to another species (or substance), the simpler or less composite it will be. From this assumption, they then go on to conclude that in any “chain” of species (or substances), ordered in terms of perfection, there cannot be any relationship of inclusion between the more and less perfect species (or substances), and no one species (or substance) less perfect than another is included in the other as a part. Finally, by then assuming that there is an isomorphism between the structure whereby species (or substances) are ordered in terms of perfection and the structure whereby individual degrees of a quality are ordered in terms of perfection, they are able to conclude that no part of one individual degree of a quality can be a part of the another individual degree, so they might as well be indivisible, and any change from one indivisible degree to another indivisible degree can only be a “complete corruption” of the first degree and a “complete generation” of or “replacement” by the second.

After providing his six arguments against (A) and (B), in the course of which he will even allow the opponent to retain their assumptions about perfection and simplicity, Scotus provides arguments against the two “simplicity” assumptions. Scotus does seem to agree that degrees of a quality come with something like a “hierarchy of perfection.” But

[32] Scotus, “Ordinatio,” interpolation to n.201; n.201.

the main challenge facing anyone who imposes such a scale of perfection on a *quality*—a challenge he believes the opponent cannot meet—is this: on the one hand, how can a quality or form become more or less without becoming *another*? And on the other hand, if each degree is distinct, *how* exactly is it distinct (without thereby destroying continuity)? A main step towards his own theory, equipped to meet this challenge, can be found in his deliberate rejection of the simplicity assumptions. Regarding the comparison with the order of perfection among diverse species (*Argument I from Simplicity*), Scotus claims:

> I say that it is to the opposite effect, because the order of species is according to quiddities and essences, and so one species does not contain the essence or quiddity of another; but the order according to degrees of the same form is according to material parts, which can exist at the same time, and the form is so much the more intense and more perfect the more it exists under several such degrees of form. [The order of perfection] exists in opposite ways, then, in this case and in that.[33]

Against the supposed similarity between accidental forms or qualities and substantial forms (*Argument II from Simplicity*), Scotus appeals to a different text of Aristotle where "in the way the more and less is asserted in accidents it is denied in substances [Meta 8.3.1043b-1044a11]," and uses this passage to sharply distinguish between how the 'more and less' in the case of substances is "according to the parts of bulk" while the 'more and less' is "according to degrees of form" in the case of qualities, where each is also expressly denied of the other:

> Hence, because [Aristotle] lays down that substantial form is in itself indivisible, therefore he does not posit one degree of form along with another; things are the opposite way in accidents, because an accidental

[33]Scotus, "Ordinatio," 1.17.2.1, interpolation n.225.

> form is divisible by way of degrees. Therefore *any degree is compatible with another degree and is perfected by it.*[34]

Scotus will thus hold both (1) that while the order of species is one of “quiddities,” and one species does not “contain the quiddity of another,” a quality or (accidental) form will be more intense and more perfect the more it “exists under several such degrees of the form”; and (2) that while in substances the ‘more and less’ has to do with the “parts of bulk” and not degrees of form, for substantial form is in itself indivisible, things are the opposite for accidental forms, for accidental form is divisible into degrees, and with regard to such qualities any degree is “compatible with another” and can even be “perfected by it.” Before looking at how Scotus develops these claims and the distinctions to which he is implicitly appealing in his own theory, let us look very briefly at some of his six arguments against claims (A) and (B), focusing on the fifth argument.

The first three arguments are directed against (A). The first argument amounts to showing that on the corruption-generation theory presented by (A), continuous changes cannot be accounted for, specifically the sort of continuity of a *habit* or *virtue* (in general) capable of increase.[35] The second argument revolves around “the perfection of that which is introduced by the increase.” The argument goes something like this: in an act that increases a habit (like a charitable act that increases charity, say), the “tenth act” can be less perfect than the “first act,” and yet with the tenth act the habit has increased to a degree that it could not have attained by the first or second act; but this sort of cumulative effect could not be if the “preexisting whole were corrupted,” because (by supposition) the perfection of

[34]Scotus, “Ordinatio,” 1.17.2.1, interpolation n.225.

[35]For details, see ibid., 1.17.2.1, interpolation in place of nn.204-5.

the first or second act was greater than that of the tenth act, and thus the individual degree of the habit generated by the first act could be more perfect than the degree generated by the tenth act. But then the fact that what is produced by the tenth act is more perfect than what is produced by the first will not be due to a "new individual" (degree) being generated by virtue of the tenth act, but rather due to the fact that something is being added to the preexisting individual generated by the preceding acts. Thus the preexisting individual would have to have remained, which contradicts the idea that with a change in degree of a quality the previous degree is entirely corrupted as the new degree is generated.[36] Thus, (A) can only be an incomplete account, since it cannot account for such changes as changes in quality and in habit that allow for "cumulative effects," but must instead suppose that whenever an agent could not already produce a (new, "higher") degree *by itself*, no action of the agent could be responsible for an increase in the degree of a quality or habit. The problem with this supposition that "then an act never intensifies a habit except to that degree which it could of itself induce"[37] is that we know by experience that it occasionally happens that "a tenth act, as equally intense as [or even less intense than] the first, intensifies the habit beyond the degree induced by the first or second act." Yet the corruption-generation account of (A) is forced to hold that, whenever there is an increase, an agent must produce the higher degree "by itself." The third argument builds on the reasoning behind the previous argument, and is an argument "taken from natural things and the action of contrary on contrary."[38]

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[36] Scotus, "Ordinatio," ibid., n.208.

[37] Ibid., ibid., n.210.

[38] Ibid., 1.17.2.1, n.212.

When a cold thing acts on a hot thing—'hotness' and 'coldness', of course, being a prime example of contraries—it "weakens" (*remittit*) the hot thing before "completely corrupting" it. When a cold object is applied to a hot object that has a certain degree of heat, the latter object will come to have lesser degrees of heat before ultimately becoming cold. But on the corruption-generation account, these lesser individual degrees must be generated *de novo* in each case.

> But I ask: by what is it generated? If one does not have recourse to a universal agent (which recourse is here irrational), no particular generator for this individual can be assigned, because the [cold] that is weakening [*remittens*] the [hot] thing cannot of itself [per se] generate an individual [hot] thing; therefore neither is the weakened [hot] thing a new individual [*individuum novum*].[39]

In other words, on (A)'s account, the only possible agent that could be responsible for the generation of these new degrees of heat would be the cold object being applied. Yet that would be to make an object, as cold, the cause of heat, which is absurd.

The fourth argument then tries to undermine *Argument II from Simplicity* by appealing to another text of Aristotle's in which he appears to deny the isomorphism between the 'more or less' in accidents and substances, claiming that the 'more or less' of accidental forms does not work in the same way as it does for substances. In the case of substances, an increase is by corruption of what preexists and by generation of the new, but this is not what happens in the case of increases of accidents.[40] The fifth argument is the most significant, for our purposes. The intent of the fifth argument is admittedly very difficult to make out, and

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[39]Scotus, "Ordinatio," 1.17.2.1, n.212.

[40]Ibid., n.213.

deciphering it will require deploying much of the heavy "Scotist" machinery. The important take-away of the argument itself appears to be that Scotus wants to argue that the particular claim of (A) that commits the opponent to holding that different degrees have no real parts in common—resting as it does on the claim that every degree of a quality is a *species* (and is *indivisible*)—will ultimately destroy any possible *quantitative* treatment of differences in degree of a quality. But this particular argument is of special importance to us, because it forces Scotus to fall back on his more nuanced, and non-standard, notion of quantity. Scotus attempts to argue against the idea that each degree of a quality is a species of that quality by suggesting that the quantitative determinations involved in determining the different degrees of a quality are not the type of thing that could determine sortal differences (as a specific difference would). In this respect, this argument also importantly anticipates the ideas involved in the formal-modal distinctions, which we will discuss in the next section.

First, Scotus reasons, one might assume that a nature that admits of 'more or less' in a determinate degree [*sub determinato gradu*] will be a species in relation to individuals. But it will then have to be an *inferior species* contained under a "species of nature." The text itself is admittedly rather dense:

> The proof of the first consequence [i.e., that a nature that admits of 'more or less' in a determinate degree will be a species in relation to individuals] is that anything that is said of individuals *per se* and quidditatively [in 'quid'], and which is 'per se one' ['per se unum'], is their species. A nature in a determinate degree—in such and such a degree—is said of individuals quidditatively and is 'per se one', because the nature in this degree [*secundum hunc gradum*] belongs essentially to the things that have nature in such a degree [*in tali gradu*], and the degree does not add to the nature something accidental to it. So it is plain that the nature in such a degree is a species, and it is plain that it is less common [*minus commune*] than the species of the nature in itself

> [*in se*]; therefore it would be a species inferior in order to the species of the nature.[41]

But one of the issues here is that in considering certain qualities, such as the color 'black' or 'white', that admit of a 'more or less' in a determinate degree (a particular shade of black say), if one regards each determinate degree or shade of black as a sub-species under the species of black, then 'black' will be the genus of such "inferior species." But if this is how all "natures admitting of a 'more or less' in a determinate degree" work, then no species of a nature capable of being intensified or weakened will be a most specific species [*specialissima*]; in particular, 'black' will not be the most specific species. Yet 'white' and 'black' are typically taken to be the "most specific species" under the genus of color. At the end of the passage cited above, Scotus includes a note in which he writes that while "a species states the 'what', a species in a certain degree states the 'what of the how much in virtue'; 'how much' is not a [specific] difference [see Ord.1 d.8 n.108]." One might hope that the passage to which he refers us, namely *Ordinatio* 1.8.3, n.108, may be of some assistance here, in helping us understand why 'how much' is not a specific difference and just what kind of determination it is. The passage in question forms a part of Scotus's influential argument for his "middle opinion" regarding how 'being' is attributed to God and creatures *univocally*:

> I hold a middle opinion, that along with the simplicity of God stands the fact that some concept is common to God and to creatures—not however some common concept as of a genus, because the concept is not said in the 'whatness' [in 'quid'] of God, nor is it, by whatever formal predication said of him, *per se* in any genus.[42]

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[41] Scotus, "Ordinatio," 1.17.2.1, n.214.

[42] Ibid., 1.8.3, n.95.

In this context, Scotus asserts that he will defend this position by defending "two middle terms (which are made clear from things proper to God): first, from the idea of infinity; second, from the idea of necessary existence."[43] The paragraphs that follow, up to and including n.108, are then devoted to the first of these—the idea of infinity. The paragraph in question reads as follows:

> [Again from the idea of infinity] From the same middle, I argue as follows: the concept of a species is not so much [*non est tantum*] the concept of a reality [*realitatis*] and an intrinsic mode of that same reality [*modi intrinseci eiusdem realitatis*], because then whiteness could be a genus and the intrinsic degrees [*gradus intrinseci*] of whiteness could be the specific differences; but those things by which something common [*commune aliquod*] is contracted [*contrahitur*] to God and creatures are the finite and infinite, which state the intrinsic degrees of it [*gradus intrinsecos ipsius*]; therefore the contracting things cannot be the differentiae, nor do they constitute with the contracted thing a composite composed in the way the concept of a species is composed, but rather [*immo*] the concept from such a contracted and contracting thing is simpler than the concept of a species could be.[44]

In looking for assistance, we seem to have run up against a limit, as if this issue involved nearly all of the key Scotistic ideas. We will thus have to discuss some more general issues in Scotus, in order to make sense of this. But before doing so, there are two main things we can already take away from the passage. First, as in the case of blackness or whiteness, where the degrees of black do not introduce multiple species of blackness so that blackness becomes a genus, so too in the case of the "contraction" of 'being', as of something "common," by the "intrinsic modes" of finiteness or infinity, this contraction does not follow the genus-species model. Thus, a difference in degree does not serve to supply sortal differences, as would a

[43]Scotus, "Ordinatio," 1.8.3, n.100.

[44]Ibid., 1.8.3, n.108.

specific difference, and this is in part what makes it "intrinsic." Second, the concept of a quality or "something common" contracted to a certain degree does not yield a *composite* concept; yet if something is a species, then its concept will be composite (namely, of genus and difference); thus, again, a quantified quality or something common with some degree—a quality or form that admits of a 'more or less'—is not to be understood on the genus-species model. The idea is that specifying 'how much' (the quantity) of a quality does not add anything over and above what is already contained in the definition of the quality itself; yet, at the same time, the different degrees of a quality lead to distinctions that are not merely mental and are "intrinsic" to that of which they are a determinate degree. We will soon be in a position to appreciate just how deeply into Scotus's thinking his ideas on the nature of quantified quality reach.

After these six arguments (the sixth of which I need not cover), Scotus claims to be able to conclude against the theory he attributes to Godfrey, and in favor of the view that "the positive reality [*realitas illa positiva*] that was in the lesser charity remains the same really [*realiter*] in the greater charity."[45] In the two interpolations in place of this last paragraph, Scotus elaborates on that conclusion, claiming that the 'lesser charity' is not "corrupted *per se*, except as to the existence that it had before," and it "remains in the [greater charity] as a part in the whole." This discussion of *parts* brings us to Scotus's own theory.

[45]Scotus, "Ordinatio," 1.17.2.1, n.225.

Scotus's Theory

Having argued against the three theories discussed above—against the view that each degree is a different species, against the view that a degree of a quality is some composite of potentiality and actuality, and against the view that a degree refers to some (in)disposition of a subject—Scotus holds that a change from one degree of a quality to another must be explained by the addition and subtraction of homogeneous parts of the quality (where 'homogeneous' means not that they are *extended* but that they belong to the same *ratio* or sort). This is what is typically referred to as his "addition theory." Scotus holds, in general, that it is not possible that change occurs without the production and/or destruction of an individual thing,[46] and accordingly agrees that in an increase of a form or quality, it is true that it is "necessary to posit some new reality, which previously did not exist, in the end term of the change; otherwise, the same thing would receive existence twice over."[47] However, he immediately qualifies this by saying that "the *whole* reality in the end term of this change is not new," for if it were, then the prior term would necessarily have to be corrupted. Thus, he reasons, it can only be that "a *partial* end term will be new (just as in generation something preexisted and something is newly acquired); therefore, in the end term of mutation something new is acquired, such that the whole is said to be new in virtue of a part."[48] Scotus has in mind here not just charity, but qualities like light, heat,

[46]See Cross, *The Physics of Duns Scotus*, 181.

[47]Scotus, *Lectura* 1.17.2.4, n.208.

[48]Scotus, *Lectura* 1.17.2.4, n.208.

and weight.[49] He also seems to have in mind arguments, such as (2) above, that showed the corruption theory's inability to account for "cumulative effects." We know that the degrees of such qualities are not composites of potentiality and actuality,[50] and that each degree is not its own species; rather, each degree will contain lesser degrees, where all the degrees are the same sort or kind of thing. Each degree of a quality is discrete, yet as an "intrinsic mode" of one and the same form or quality, all such degrees are degrees of precisely the same sort of thing—and so they are included in the continuum of that form or quality.[51] Moreover, not only does Scotus not regard a quality quantified with a certain degree on the model of genus-species, but to the extent that one can speak of species of quality, a quality is indeed held to be *divisible* into distinct quantitative degrees. But if each degree is *distinct*, and if a change in degree will correspondingly entail a change in identity (at least in one sense of *identity*)—something Scotus does hold—how can this theory fare any better in terms of securing the continuity of alternation-type changes? To see this, we must get a better understanding of Scotus's notion of an "intrinsic mode" and the sort of identity or unity it supports, notions the development of which reached very deeply into many areas of Scotus's thinking.

Stepping Back: Scotism More Generally

One of the principle motivations behind Scotus's notion of an "intrinsic mode" involved the re-conceptualization of infinity. For many thinkers of this period (for instance, Aquinas),

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[49]See Scotus, *Lectura*, ibid., n.120; see also Cross, *The Physics of Duns Scotus*, 187.

[50]Scotus, *Lectura* 1.17.2.4, n.196.

[51]See Scotus, "Ordinatio," 1.17.2.2, n.252.

still faithful to the Ancient Greeks, infinity was understood negatively (in terms of *a-peiron*). More specifically, it was often taken to be a (negative) relational property, since an infinite being, by definition, was said to lack any relation to an entity capable of limiting it. This negative conception stood at the base of the two other main medieval developments of the concept: what I call, in the final chapter, the economical-prismatic infinite and the tragic infinite. For Scotus, on the other hand, by thinking of infinity as an "intrinsic mode," he is able to understand it as a *positive* reality.[52] Scotus grounds this approach on what he calls a "transferred" sense of magnitude, still thoroughly quantitative but now meant to cover non-extensional differences in degree of a single reality. There are a few angles from which to approach this move, but it is perhaps best understood after being placed in the context of Scotus's defense of the primacy of a univocal (as opposed to analogical or equivocal) concept of being (*ens*). I will thus begin with a brief discussion of this.

Univocal Being

Against a long tradition—starting perhaps with Aristotle's analysis of the *pros hen* relationship and culminating in Henry of Ghent's theory of being-as-analogical—Scotus argues that the concept of being is *common to* God and finite (created) beings, as well as to substance and accident (and more broadly, the ten categories, and a number of other disjuncts). Initial evidence for the plausibility of this thesis, according to Scotus, is to be found (i) in the fact that analogical reasoning would seem to tacitly require, as its basis, that there be common concepts; and (ii) in the fact that we seem to be able to doubt, for instance, whether God

[52]Historically, this seems to prepare for the conception of infinity as plenitude (that reaches its pinnacle in the time of Spinoza and Leibniz), and to a lesser degree, for the notion of infinity as the complication of contraries (as found in Cusa).

is infinite or finite, while remaining certain that God is a being. For Scotus, this latter fact suggests that the concept of being, while truly predicated of both finite and infinite being, is itself "indifferent," and so univocal, to both. In the most general sense, a univocal concept is defined as "a concept that is one in such a way that its unity suffices for a contradiction to be generated when it is affirmed and denied of the same thing."[53] Common to the ten categories—applying to all sorts of realities regardless of kind—'being' is held to be a *transcendental*.[54] A transcendental contributes significant distinctions to reality yet is trans-categorical and thus "not contained under some one genus."[55]

As the most common concept, the proper or primary object of metaphysics is thus being *qua* being—not God, not substance. In the widest of senses, being is whatever does not include a contradiction. But more narrowly, some beings are held to be more complete or 'perfect' than others, yielding an "order of perfection."[56] According to Scotus, in considering various concepts and resolving composite concepts into simpler ones—ultimately arriving at primary ones that cannot be analyzed or decomposed further—one must bottom out on

[53]Scotus, "Ordinatio," 1.3.2, n.5.

[54]See ibid., 1.8.1.3.

[55]The so-called "proper attributes," as well as the "disjunctive attributes" and the "pure perfections," are also held to be transcendentals. Briefly, the proper attributes were features including its subject in its definition, but not conversely. For instance, odd is a proper attribute of number, since odd numbers include number in their definition, but number does not similarly include oddness. 'One', 'true', and 'good' are given by Scotus as examples of three proper attributes of being. While these are coextensive with being, each adds something distinctive to the concept of being—this is why being cannot be predicated *in quid* of its proper attributes. The disjunctive attributes, for their part, were held to be coextensive with being, immediately predicated of being, and dividing it completely, such as with the disjuncts 'infinite-finite,' 'necessary-possible'. Finally, the "pure perfections," on the other hand, are not coextensive with being. These are "better to have" than to not have; they are all compossible with one another, simple, and compatible with infinity; they are equally perfect, and no pure perfection is "formally unsharable" or incommunicable.

[56]See Scotus, "Ordinatio," 1.3.qq.1-2.

concepts that are "simply/absolutely simple" (*simpliciter simplex*). But there is only one such irreducibly simple concept: the *quidditative* concept of being. There are also those concepts that give ultimate differences (where "ultimate" means such differentia do not themselves admit of differentia), for instance: individuating difference, specific difference, generic difference (and still more broadly, the transcendental differences qualifying being). But if being were univocally predicated of such ultimate differences (or the proper attributes) *essentially*, i.e., *in quid*, we would either enter an infinite regress, or the ultimate differentia would lose their alleged simplicity. Thus 'being' is not univocally predicable *in quid* of the ultimate differentiae or of its proper attributes. All these transcendentals are thus held to be modifications of being (*passiones Entis*)—for being is univocally predicable of them not *in quid*, but *in quale*. Yet it is equally important to realize that since proper attributes characterize all beings, and ultimate differentia "constitute" beings, such concepts must still be accorded the highest ontological status. Scotus attempts to account for both of these aspects with his notion that these other transcendentals are "virtually" contained in the concept of being.

The main issue in all this can be summed up thus.[57] Two things are *different* when there is a real common factor together with a real distinguishing element. This is what happens with different species sharing a proximate genus, but distinguished from one another by differentia of that genus. On the other hand, two things are *diverse* when there is no real common factor shared by them, and so no way of supporting a distinguishing element. This is what happens with the ten categories, held to be diverse from one another. Likewise, God

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[57]This paragraph basically follows King, "Scotus on Metaphysics," 7-8.

and creatures, infinite and finite, were traditionally held to be diverse, since there appeared to be no reality common to them. But Scotus can be seen as attempting to reduce what was traditionally held to be a diversity among the categories and between God and creatures to a *difference*. While we have begun to see Scotus's general strategy for getting around the problem that if 'being' were made univocally predicable *in quid* of the ten categories, then it would have to be the supreme genus above them all, the other side of the issue is that by making God and creatures different and not diverse, there will clearly have to be some real factor common to God and creatures. This would not only challenge the alleged transcendence of God and the gap between the latter and created beings, but it would also seem to entail that God could not be *simple*, for it would seem to require that God was a "real composition" of common and differentiating factors. For our purposes, the most significant aspect of these problems is how, in insisting on the univocity of being, Scotus answers the challenge of ensuring that God's simplicity is not thereby undermined; for it is in attempting to avoid such undesirable conclusions that Scotus develops some of his most decisive ideas and distinctions. Let us look more closely at how Scotus does this.

According to Scotus, inquiry into God typically begins with consideration of a "formal perfection" attributed to God. We then form a conception of *ens infinitum* by attributing those same properties and getting rid of any limitation in their notion. But there is another way. We can attribute *ens* (being) "in an unrestricted manner," arriving at the concept of *ens infinitum*, which will "virtually contain" all the "pure perfections" of the first way. This second way thus arrives at a "simpler" concept, the concept of an infinite being, which is simpler than concepts like "good being" or "true being," for infinity is not an attribute or property of 'being'. Rather,

> it signifies an intrinsic mode of that entity, so that when I say "infinite being," I do not have a concept composed accidentally, as it were, of a subject and its attribute. What I do have is a concept of what is essentially one, namely of a subject with a certain grade of perfection—infinity. It is like "intense whiteness," which is not a notion that is accidentally composed, such as "visible whiteness" would be, for the intensity is an intrinsic grade of whiteness itself. Thus the simplicity of this concept "infinite being" is evident. Now the perfection of this concept is proved first from the fact that it virtually includes more than any other concept we can conceive. As "being" virtually includes the "good" and the "true," so "infinite being" includes the "infinitely good," the "infinitely true," and all pure perfections under the aspect of infinity.[58]

It seems plausible that in the concept of infinite being, 'infinite' does not act as an attribute or property, and so the concept of infinite being would not be "accidentally composed" of a subject and attribute, as something like "visible whiteness" would be. But what does it really mean to claim that this concept designates what is "essentially one, namely a subject with a certain grade of perfection," i.e., designates a certain type of unity (that admits of degrees)? And what does it mean that, on the model of intensities of qualities or forms such as "intense whiteness," infinity is an "intrinsic mode" of being? What exactly does it mean to say that it is "intrinsic"?

We can begin to understand this by considering that to which "intrinsic" is being opposed. Scotus claims that if you understand "highest," for instance in the phrase "highest good," to be taken in a *comparative* sense,

> then it includes a relation to something extrinsic to the being[. . .]. But if "highest" is understood in an absolute sense, i.e., as meaning that the very nature of the thing is such that it cannot be exceeded, then this perfection is conceived even more expressly in the notion of an infinite

[58] Scotus, "Ordinatio," 1.3.1.1-2, n.58.

> being, because "highest good" does not indicate as such whether it is infinite or finite.[59]

But once one wants to attribute such a "grade of perfection" both "intrinsically" and "absolutely" to being, many questions arise regarding the fact that this account seems to be invoking some notion of *quantity* or *quantified being*, yet it is not clear whether the traditional category of quantity can handle an "infinite grade" or any of the other demands being placed on it. It is in this context that Scotus introduces his notion of a "transferred" sense of magnitude, no longer "a quantity of bulk [*quantitas molis*], but rather of power [*virtutis*]."[60] Scotus is willing to concede that no difference in the category of quantity belongs to God, nor does any property in that genus belong to him; rather, "a transcendent property does so belong."[61] Many of Scotus's arguments for an *actually* infinite being (and the primacy of the infinite-finite disjunction) rest on the development of this transcendental or trans-categorical sense of quantity. After all, while the disjunction infinite-finite is the "first of the disjunctions" of being, strictly speaking

> finite and infinite are immediate divisions not of being but of "quantified being." For just as finite and infinite, according to the Philosopher, correspond to quantity (which is true of "finite," "infinite," and "quantity" proper), so also, *in an extended sense*, finite and infinite as proper [disjunctive] attributes of being pertain precisely to a "quantified" being in the sense of having in itself some amount of perfection. But this sort of "quantity" pertains only to entity that can be "total" or "partial" in the hierarchy of essences. For one "quantity" compared with

[59]Scotus and Adams, *Duns Scotus - Philosophical Writings*, 28.

[60]Scotus, "Ordinatio," 1.19.1.2, n.18.

[61]Ibid., 1.19.1.2, n.24.

> another must immediately excel or be excelled and must have partial or total being.[62]

Regarding this "hierarchy of essences" ordered according to their magnitude of perfection, while it is true that for "limited natures" it is necessary for the "greatest perfection they are capable of" that several such natures be ordered by "an order of unequal perfection," such an order is not necessary for "the greatest perfection, because that can exist in the most perfect unlimited nature, without an order of imperfection."[63] Shortly after bringing up this comparison of one "quantity" (in an extended sense) that, as a "quantity of perfection," immediately excels or is excelled by another in the "hierarchy of essences," Scotus elaborates on this "extended sense" of magnitude:

> we have concluded that in the divine there is something extra-mental possessing magnitude, namely the essence, and that its magnitude is something real and extra-mental. We also know what sort of magnitude it is[...]; *it is magnitude not in a proper but in a transferred sense.* In Augustine's terminology, it is not magnitude of bulk but of perfection. The divine essence would still retain its own magnitude or proper infinity, even though—if this were possible—it were stripped of all its properties. For it has its own intrinsic degree just as something finite has its own finitude. Man, for instance, even if one abstracts from all his properties, still retains his essential finitude in the hierarchic classification of beings. In this sense, then, I concede that "to be" and "to be great" are identical in God. And there is a sense also in which it is correct to say "to be great" is even more intrinsic to God than is "to be just" or "to be wise," for "great" does not express a property or attribute as "just" and "wise" do. Greatness, therefore, is indeed "fused with him" because of the supreme identity of magnitude and essence. But when you infer from this that magnitude no longer exists there in any proper sense of the word, the implication must be denied. *Even*

[62]Scotus and Adams, *Duns Scotus - Philosophical Writings*, 129, my emphasis.

[63]Scotus, "Ordinatio," 1.19.1, n.13.

> *in creatures magnitude is so fused with its perfection that it does not represent something really other than that of which it is the degree*[...].[64]

This "supreme identity" or "fusion" of *magnitude in a transferred sense* and essence, leading to the notion of a being (which includes both God and created beings) that is "so fused with its perfection that it does not represent something really other than that of which it is the degree," is one of Scotus's most far-reaching ideas. It even inspires him to challenge the standard account of the Aristotelian infinite as the *potential infinite*, i.e., where "no matter how much one removes from it, there is always more for the taking," from which notion it could only be inferred that a being infinite in this manner would not fulfill the notion of a *whole*, for a whole has nothing outside itself, nor could it be perfect, "for the perfect lacks no perfection, whereas this always lacks something."[65] In order to remedy such deficiencies, Scotus proposes an alternative to the standard conception of the infinite as potential, a wager that is deeply connected with his efforts to develop the concept of an intrinsic magnitude or intensive quantity:

> For our purposes, let us change the notion of the potentially infinite in quantity, if possible, to that of the quantitatively infinite in act. For just as it is necessary [in the case of the potentially infinite] that the quantity of the infinite should always grow by receiving one part after another, so we might imagine that all the parts that could be taken were taken at once or that they remained in existence simultaneously. If this could be done, we would have in actuality an infinite quantity, because it would be as great in actuality as it was potentially. And all those parts which in infinite succession would be actualized and would have being one after the other would be conceived as actualized all at once. Such an infinite in act would indeed be a whole and in truth a perfect whole, since there would indeed be nothing outside it, and it

[64] In Alluntis and Wolter, *God and Creatures*, 138-9 (The Quodlibetal Questions, 6.1.3), my emphasis.

[65] In ibid., 108-9.

> would be perfect since it lacks nothing. What is more, nothing in the way of quantity could be added to it, for then it could be exceeded.[...]
>
> From the notion of the infinite in the *Physics*, then, applied imaginatively to something infinite in quantity, were that possible, and applied further to something actually infinite in entity, were this possible, we can form some sort of idea of how to conceive a being intensively infinite in perfection and power. This enables us to describe a being infinite in entity as a being which lacks no entity in the way that one single being is able to possess it. The qualification "in the way, etc." is added because a single being cannot possess every entity whatsoever formally and by a real identity.[66]

In order to better unpack the final sentence of this passage, and to start tying together all of these reflections surrounding the "transferred" sense of quantity, we will now look more closely at Scotus's formal and modal distinctions and the various senses of "identity" generated by these distinctions. This will, in turn, give us a better sense of Scotus's efforts to rethink various natures on the model of certain forms and qualities that are continua-like.

Formal and Modal Distinctions

As both the formal and the modal distinction are meant to be more fine-grained than the extremes of the real-conceptual distinction, we should begin by considering Scotus's explanation of the extreme or limit case of "real distinction" (or conversely, "simple identity"). For Scotus, two entities are *really distinct* whenever it is possible ("at least by divine power") that the one exist even if the other does not.[67] Conversely, two things will be *really or simply identical* if and only if they are not really distinct, i.e., if and only if neither can possibly ("even by divine power") exist without the other.[68] For Scotus, however, this rather crude

[66] Alluntis and Wolter, *God and Creatures*, 110.

[67] See Scotus, "Ordinatio," 2.2.1.2.

[68] See ibid., 2.1.4.

distinction (and its corresponding identity) is by no means the end of the story, for really identical things are not necessarily totally the same (or the same in *other senses of same*), for they can have distinct *formae* (or *realitates* or *formalitates*).[69] Scotus usually deploys the formal distinction in his discussions of perfections, which includes things like "wisdom" or "will."[70] But while "less than the real distinction"—since only items that are really identical can be distinguished formally—formally distinct items are *not* distinct merely on account of an activity of the intellect. Rather, this distinction is "on the side of the thing" (*parte rem*). The formal distinction thus describes a distinction weaker than real distinction, yet not a merely conceptual distinction. The formal distinction will only properly apply "within" the boundaries of a given single real thing. Formally distinct items differ in their *ratio*. As King points out,

> there is a formal distinction between each of the following (within a given individual thing): the genus and specific differentia; the essence and its proper attributes; the faculties of the soul, and the soul itself; the Persons of the Trinity, and the divine essence; the uncontracted common nature and the individual differentia—and this list is not exhaustive.[71]

As before, the converse of this distinction gives the corresponding idea of *formal equality* (occasionally called "likeness").

On the face of it, the very admission of distinct formalities would seem to threaten to make composite or non-simple the being to which they apply. This would obviously pose a problem for God's alleged simplicity. But it is in fact this very notion that was meant, at

[69]In other words, for Scotus, the "Indiscernability of Identicals" fails.

[70]Scotus, "Ordinatio," 1.3.1.1-2, n.39.

[71]King, "Scotus on Metaphysics," 10.

least in part, to preserve God's simplicity. Scotus's formal distinction, while being stronger than a mere conceptual distinction, is designed so as not to have to give up on simplicity. For formally distinct formalities make up a unity, though they do not do so through "real composition."[72] Moreover, such a form, when considered without its intrinsic modes (its degree of perfection), is both simple and common. Then it is "contracted" to an infinite degree in God, and to a finite degree in all else.

This notion of contraction to some degree and the talk of "intrinsic modes" brings us to the modal distinction and the idea of "modalization."[73] For Scotus, "If an entity is finite or infinite, it is so not by reason of something incidental to it, but because it has its own degree of finite or infinite perfection respectively."[74] Wolter accordingly defines an intrinsic mode as a

> qualification so identified with the subject it modifies that it is neither really nor formally distinct from it, yet it is possible to conceive the subject without the mode as a first intention at the level of abstract cognition.[...] As an intrinsic mode, magnitude is the degree of intensity or measure of intrinsic excellence characteristic of some formal perfection as it exists extra-mentally in a particular subject.[75]

It is only with the modal distinction that Scotus is able to most satisfactorily resolve the issue of preserving God's simplicity and unity while also acknowledging the many irreducible

[72]See King, "Scotus on Metaphysics," 11.

[73]Whatever it ends up being, "modalization" is certainly *neither* instantiation *nor* differentiation: "The relation between a 'whiteness in the tenth grade of intensity' and whiteness itself is not that of instantiation, since whiteness itself could never exist as such but must always exist as some share of whiteness"; moreover, whiteness in the tenth grade of intensity is also "not a species of whiteness," which rules out differentiation" (King "Duns Scotus on the Common Nature," 16).

[74]Quoted in Cross, *Duns Scotus*, 40.

[75]See Alluntis and Wolter, *God and Creatures*, 508.

and more-than-conceptual differences within natures. Modal distinction involves a "lesser" distinction than the formal distinction, but again, this is a distinction that is not merely conceptual. Moreover, a degree of intensity of some form or quality is not a differentia of the genus to which that quality belongs, as whiteness to color. Yet as "contained within a given form," these modes are "intrinsic": they are not outside the *ratio* and as such they are neither a distinct ratio nor something added "extrinsically" to it. It is thus the case that while the formality does remain identical through changes in mode or intensity, in an important way, the most demanding sense of the *nature* of a thing will be inseparable from its degree of intensity. While we can indeed conceptually separate whiteness from the particular degree of intensity with which it exists, Scotus argues that in doing so our conception of the entity is no longer perfect or entirely adequate to the nature in its most complete sense.

All of this—including how formally distinct formalities are unified under a single really identical entity and how modally distinct intensities are unified under a single formality, without either unity becoming one of "real composition" of the distinct items or degrees that fall under that unity—is closely related to Scotus's subtle treatment of the *continuum*, an issue that also appears to have motivated his turn to "intrinsic modes" in the first place. As we saw in the previous chapter, Aristotle held that no continuum can be *composed of* indivisibles or 'points'. As one would expect, Scotus is also opposed to atomism, and so would not hold that a form or quality, as a continuum of sorts, was *composed* of its intensive modes or degrees. But Scotus does hold that these degrees are *unified* under one formality, and he moreover argues against "non-entitism": the theory that indivisibles do not exist at all. He argued that indivisibles do exist, for, like Aristotle, while Scotus holds that no continuum can be *composed of* indivisibles (whether these are extended or unextended), this does not negate the fact that continua have limits and these limits are indivisible. While the standard approach emphasizes the infinite divisibility of continua, so that any of its parts is separable from any other parts, again like Aristotle, Scotus emphasizes that a continuum must also be seen to be *one* thing, which requires a strong account of how the parts are *united* to form a whole.[76]

While I do not have space to consider Scotus's discussions of the continuum, these sorts of questions surrounding the continuum are very closely related to the issues of the different types of identity corresponding to the different distinctions (real-formal-modal). Relating the three types of distinction or non-identity—real distinction, formal distinction, modal distinction—we have that the formal distinction is "less than" the real distinction, while the modal distinction is "less than" the formal distinction. Modal distinctions are distinctions *within* a single formality, and formal distinctions are distinctions *within* a single thing. An intrinsic mode or magnitude is not distinguished from another mode in the way distinct formalities are distinguished from one another (nor, of course, in the way really distinct entities are distinguished from one another), and distinct formalities are themselves not distinguished in the way really distinct entities are distinguished. Moreover, since distinct degrees of the same formality are distinguished only modally but not formally or really, they cannot be distinguished as distinct things or as distinct *relations* to the form, but must be distinguished as greater or lesser perfections of that form.[77]

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[76]For a discussion of how Scotus treats these questions of the continuum, see Cross, *The Physics of Duns Scotus*, 135.

[77]Scotus, "Ordinatio," 1.19.1.1, n.17.

In terms of the corresponding notions of equivalence or unity, there are three types: namely intensive equivalence (modal equality), likeness (formal identity), and real identity. Moreover, we have that the equivalence or unity of modally equivalent degrees is a stronger type of equivalence than formally equivalent formalities (for the former are equivalent in degree and, by definition, within the same formality, whereas the latter need not be equivalent in degree), while formally equivalent formalities gives a stronger type of equivalence than real identity (for formally equivalent formalities are both one in formality and in the same thing, whereas really identical things need not have the same formalities). For this reason, since *modal equivalence* is the strongest of the three types of equivalence or identity, Scotus argues that identity or equivalence in intensive modification is the most *perfect* or *complete* kind of unity.

We know that essences (quiddities) contain formal natures (*rationes*). And changes at the modal level, variation in intensity, cannot directly alter the essential order—redness remains redness through changes in intensity. And of course, modes themselves do not have an essence—for what has a quiddity must be able to be conceived *per se*, but modes can only be conceived together with the formal natures of which they are the modification. We are thus faced with an important question: how can Scotus speak of the "supreme fusion" of intensive magnitudes and essence? The answer lies in the distinction between how quantity in the standard sense—as quantity of bulk—works as against his transfered sense of a quantity of perfection or intensive magnitude:

> although a quantity of bulk states something added to the nature of the subject, and therefore it cannot remain under its formal idea and also pass over into the essence by identity, yet magnitude of power in every being passes over into that which it by identity belongs to, even

> in the case of creatures. Proof: for if an angel has some magnitude of power (about which Augustine speaks in ibid. VI ch.8 n.9: 'In things that are not great by bulk, what it is to be greater is to be better'), and if its perfectible magnitude is not the same as its essence, let it be removed from the essence. With the essence then remaining, I ask what grade of perfection it has among beings? For it will be nothing unless it has some determinate grade of perfection among beings; therefore there still remains in the essence a magnitude of power, whereby it is said to be thus or thus perfect. Therefore the quantity in everything passes over by identity, and remains in everything in its proper idea, because the nature of such quantity is to state the intrinsic mode of the perfection it belongs to; and from the fact that it states 'mode', it remains; but from the fact that it states 'intrinsic', it passes by identity into the essence to which it belongs.[78]

Another approach to this "supreme fusion" of intensive magnitudes and essences might be to recognize how Scotus will even go as far as to assert that an "equality of power" between two entities, their ability to act on other extrinsic things, is ultimately rooted in "something absolute and intrinsic to them," i.e.,

> from the fact that some beings have some form to an equal degree it follows as a consequence that they can act upon extrinsic things to an equal degree. Equality of power, therefore, is not, properly speaking, distinct from the equality of magnitude, but it represents, as it were, an explication of the kind of equality in magnitude, namely the characteristic of an active form. Thus creatures which possess heat in equal magnitudes have equal power to heat.[79]

But we also begin to see the real force of this "supreme fusion" when he discusses how with the most fine-grained distinction of all—the modal distinction—we gain access to the most "perfect" or "adequate" concept of certain realities:

> [W]hen some reality is understood along with its intrinsic mode, the concept is not so simply simple that the reality cannot be conceived

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[78] Scotus, "Ordinatio," 1.31.1, n.10.

[79] Alluntis and Wolter, *God and Creatures*, 132 (The Quodlibetal Questions, 6.1.2.8).

> without the mode, but it is then an imperfect concept of the thing; the concept can also be conceived under that mode, and it is then a perfect concept of the thing. An example: if there were a whiteness in the tenth grade of perfection, however much it was in every way simple in the thing, it could yet be conceived under the idea of such an amount of whiteness, and then it would be perfectly conceived with a concept adequate to the thing itself, or it could be conceived precisely under the idea of whiteness, and then it would be conceived with an imperfect concept and one that failed of the perfection of the thing; but an imperfect concept could be common to the whiteness and to some other one, and a perfect concept could be proper. A distinction, then, is required between that from which a common concept is taken and that from which a proper concept is taken, not as a distinction of reality and reality but as a distinction of reality and proper and intrinsic mode of the same, which distinction suffices for having a perfect or imperfect concept of the same thing, of which concepts the imperfect is common and the perfect is proper.[80]

Certain formalities are regarded as supporting a range of intensities and as being that which unifies them without thereby being *composed* of distinct degrees—in this way, such formalities are treated as *continua*. The nature of such formalities and the modal intensities that modify them provide Scotus with a model that he then uses to re-shape his vision of all being. While originally inspired by examples of accidental forms and the sort of continuity that appears to exist for changes in qualities like redness or charity, Scotus insists that the "transferred" sense of quantity is not like the sensible qualities to the extent that they are accidentally composed. One can indeed separate these formalities, by an act of the intellect, from any of their degrees of intensity, thus arriving at a concept of the underlying reality that is, on the one hand, "more imperfect in being less adequate to the thing itself" (for the thing is fused with its magnitude), and, on the other "more common" (for in regarding it independently of its degrees, one grasps how it is common to various degrees of perfection

[80]Scotus, "Ordinatio," 1.8.1.3, nn.138-140.

supported by that formality). But we should not let the fact that the most "adequate" concept of certain realities is one that conceives that reality under a determinate mode, i.e., via the modal distinction, blind us to the equally important phenomenon of *commonality* (something that is dimly indicated by our imperfect *concepts* of something as common). While certain formalities are "contracted" to determinate degrees, degrees that are "intrinsic" to them, we can nevertheless glimpse the "indifference" of such formalities or realities to determinate grades or degrees. While these formalities remain intervals of intensity, such "indifference" of these formalities to their determinate degrees, manifests itself in the fact of the "commonness" of certain natures, or, at the level of our thinking, in what Scotus calls "common natures." Before turning to Oresme, we will end this section on Scotus with a closer look at Scotus's notion of *common natures* (*natura communis*), together with a discussion of how this connects to his formal-modal distinction and what these connections reveal about Scotus's take on the continuity-generality connection.

Common natures

Scotus holds that it is not true that everything is either universal or individual. Something (such as a formality or reality) can be neither universal nor individual, "indifferent" to both of these. Such a "nature" is a principle of unity that is "less than numerical," and as such is something that is meant to account for that nature's being the *kind* of nature it is.[81] One of Scotus's strongest arguments for the existence of such a less-than-numerical unity is this:

> If every real unity is numerical unity, then every real diversity is numerical diversity. The consequent is false. For every numerical diversity, in so far as it is numerical, is equal. And so all things would be equally distinct. In that case, it follows that the intellect could not abstract something common from Socrates and Plato any more than it could from Socrates and a line. Every universal would be a pure figment of the intellect.[82]

In other words, if things only differed numerically, then it would follow that all things would have to be equally distinct; yet this is not the case, for we observe degrees of distinction, e.g., between Socrates and Plato vs. between Socrates and a line vs. between a line and a color. If things were all equally distinct, this would moreover destroy the foundation of universality.

The "nature" that is this less-than-numerical unity is to be distinguished from the standard 'universal', which Scotus speaks of as the "logical universal" or the "universal in act." The common nature (or better: just "nature"), on the other hand, is the *fundamentum*

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[81]Wolter notes how in the Scotistic literature such a quiddity or nature is "incorrectly referred to" as common nature. This is "incorrect" since this "commonness" is just how the nature appears once conceived abstractly by the intellect: "As a "nature," however, it rather has the character of being "indifferent" either to being present in the individual or, when abstracted by the mind, to being "predicated of many" (Frank and Wolter, *Duns Scotus, Metaphysician*, 197). As should be clear from the remainder of my discussion of this concept, I fundamentally agree with this observation, however I occasionally follow the tradition of referring to "common natures" where I mean "natures" (the thought of which makes them "common natures"); but the warning by Wolter should be noted. It is also worth noting how, in being *real* without being determined to *exist* in any given thing, in lacking a numerical unity, the common nature is not "common" in any ordinary sense. In fact, even if there is only one human being left, this person will still have a common nature. In short, the notion does not depend on anything like an instantiation across many particulars or any sort of collectivity; it is not identical with a numerical one nor is it a plurality of numerical ones. As Grajewski writes: "[the common nature is] a certain nature which is ordinarily referred to as common or universal nature but which, in reality, is not common, nor universal, nor particular" ("The Formal Distinction of Duns Scotus," 143).

[82]Scotus, "Ordinatio," 2.3.1.1, n.23.

universalitatis,[83] the ground or basis for universality, the real commonness of a nature that does not reduce to a logical question of predicability but forms the ground for the very possibility of predication.[84] While it is a fact that there are relations of "similarity" between various individuals, these relations are not inexplicable or self-evident—rather, there must be an account capable of *explaining* the "real foundation" of such similarities.[85] If it is a fact that certain things, such as X and Y (e.g., red and magenta) are similar in a way that X and Z (e.g., red and a line) could never be, then if it is not true that what accounts for this difference in the pairs is something that can be found at the level of the "individuals," then it must be in some sense "real" without being another "individual thing"—thus it must be general. This general nature is quite distinct from the traditional, logical universal in that, unlike the traditional universal, if every individual were destroyed, the nature would thereby be destroyed.[86] Also, if Plato is a human, the idea seems to be that there must be something "in" Plato that can ground this claim. Yet the "common nature" itself is neither a *suppositum* nor something separate (like a Platonic form).

Scotus's ideas on the "nature" (or common nature) seem to have largely emerged out of a remark made by Avicenna in his *Metaphysics* V.I, namely that "horseness is just horseness—it is of itself neither one nor many, neither universal nor particular."[87] Scotus likewise holds the common nature to be "of itself naturally prior to" being one or many, uni-

[83]Scotus, *Metaphysics*, 7.18, n.6.

[84]Of course, not all predicates imply a real commonness, but some do.

[85]Scotus, "Ordinatio," 2.3.1.1, n.18.

[86]Scotus, *Metaphysics* 7.13, nn.2-3.

[87]Cited and discussed in Scotus, "Ordinatio," 2.3.1.1, n.31.

versal or particular; yet it "never really *exists* without some of these [features]."[88] Thus, the common nature "of itself"—that is, as "uncontracted"—is neither universal nor particular, neither one nor many; prior to being "contracted" by such features, it is *indifferent* to them, as it may take on such features or fail to have them, while remaining what it is.[89] However, by virtue of being general, and by virtue of serving as the real foundation for universality, the commonality of common natures is still closely related to issues of universality, even as the former is more a question of the structure of reality and less one of our concepts of that reality. Scotus says that while "community belongs to the nature apart from the intellect...[and so] community belongs to the nature in its own right [...], universality does not belong to a thing in its own right."[90] In other words, for Scotus, there is no real universality, yet this does not imply that there are only individuals—for there is real commonality, which in turn serves as the foundation for universality (which is inherently *mental* or a question of our concepts).

For Scotus, perhaps most importantly, the uncontracted or "indifferent" nature is not related to the "contracting" differentia as the genus is related to the specific differentia, but rather as "a reality to its intrinsic mode." In other words, the "uncontracted nature," itself the real ground of universality, is related to the contracted nature as a reality or formality is related to an intensive mode. But it is equally clear that the uncontracted nature is

[88]Scotus, "Ordinatio," 2.3.1.1, n.32.

[89]As for one side of this "indifference": since the common nature is not of itself determined to any particular individual, there must be a principle by which the common nature is realized by individuals—this is of course supplied by Scotus's famous "principle of individuation," or *haecceity*. The point is that the common nature can be "universalized" in the mind or "contracted" in an existing individual—without it thereby collapsing into either of these two extremes (i.e., the universal or an existing 'this').

[90]Cited in *Cambridge Companion to Duns Scotus*, 110.

indifferent to differentia, in the same way a formality can just as well be “modalized” by one intensive mode as by another in the allowed interval of intensity, and so remains “indifferent” to these determinations. King remarks that it is precisely this “indifference” that “is what the commonness of the common nature amounts to.”[91] In other words, what is common in such “natures” has the sort of continuity a continuum has before it has been cut up into parts. Yet since the nature is always completely contracted in every existing individual, just as any given formality will always be determined to a definite degree, it is not immediately clear how this idea of a (common) “nature” as “indifferent” is supposed to give us something like a real unity or commonness over and above mere conceptual commonness. King’s answer to this seems to be that “The actualization of a common nature by an individual differentia does not “use up” the real potencies belonging to the nature, which are retained even while contracted.”[92] In other words, the real commonness lies in something like “real potencies” inherent in the uncontracted nature, potencies that are not only “prior” to their modalization in an individual but are not eliminated with the modalization of a nature in some definite degree.

> [T]he possession of these real potencies is a property of actual things; these potencies provide a certain kind of real unity, though not as tight a unity as numerical unity would be—it is a real unity that is less than numerical unity.[93]

It would take us too far afield to fully evaluate this interpretation, so I will confine my attention to a separate observation. If the “nature” (or common nature) embraces an entire

[91]King, “Duns Scotus on the Common Nature,” 22.

[92]Ibid., 22.

[93]Ibid., 23.

spectrum of degrees but is always individuated in an existing thing, what does it actually mean to say that the common nature is not itself a thing, a *res*, nor a universal, yet somehow it has a status apart from the mind? Here is where we see the full force of the formal and modal distinctions, together with the fact that the transcendental sense of quantity generates a ranking or ordering of essences. Since the first division of intrinsic modes is into infinite and finite, whereas other divisions—including contingency-necessity, existence, haecceity—are posterior, the ordering of beings according to their "degree or quantity of perfection" is held to precede any ordering or determination of beings provided by individualization, existential instantiation, or necessity. According to Scotus, any being has three aspects: "I say that any being in itself is a 'what', and has in itself some determinate degree among beings, and is a form or has a form."[94] The unity characteristic of common natures is not a unity of *really identical* entities. Nor is it a unity of *modally equivalent* degrees of a given formality or reality. Rather, the unity of a common nature is that which belongs to a formality or reality in so far as it can be modified in many ways, passing through various intensities within a given range, i.e., in so far as certain realities behave like *continua*. It is significant that such natures are held, by Scotus, to be the real foundation for universality. It is no coincidence that, emerging out of his reflections on contrariety, alteration, and continuous changes, in his efforts to rethink how certain natures and formalities are continua-like, we see Scotus attempting to forge connections with generality, or the real foundations of universality.

One of the problems with Scotus's account, admittedly, is that he does not of course develop anything like an account of what it might actually look like for different formalities

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[94]See Scotus, "Ordinatio," 1.19.1.1.

to be modified with certain intensities *in different ways*. This means that, ultimately, the intensive modifications of formalities will be rather monolithic. Scotus thus offers no help to those who might want to fruitfully *compare* the different measures or scales that span the intensive modifications of *distinct formalities*. It is precisely in this respect that Oresme advances beyond Scotus's account. Scotus could be thought of as providing the general schematics or *algebra* of intensities. From certain qualities admitting intervals of continuous variation or degrees, Scotus makes the bold extrapolation to all essences having an "intrinsic" grade or degree of perfection, and to formalities as having a kind of (non-accidental and non-mental) unity that supports (modal or intrinsic) distinctions in degree. In this way, Scotus initiates serious consideration of the possibility that there are many more unities or natures than the tradition had been willing to accept (beyond those found in the study of magnitudes) that, in supporting a range of degrees or internal variability (all without thereby losing its unity), act like *continua*. Moreover, since Aristotle, nearly every philosopher had held discrete quantity to be the "real" sort of quantity, next to which continuous quantity had something of a shadowy, uncertain existence; with Scotus's "fusion of essence and magnitude of perfection," and his notion of certain realities supporting a range of intensive modes (and their "fusion" with such degrees as giving the most "complete" concept of them), Scotus begins to reverse this traditional preference for discrete quantities. Finally, in taking seriously that there were certain natures that admitted of degrees yet remained a unity, and had a less-than-numerical unity, Scotus also began to take more seriously that there were generalities (in the form of "indifferent natures" or common natures) that both "behaved like" continua and were the real foundation of universality (and so, were non-conceptual). Oresme's efforts to construct a "configuration theory" that begins to allow one to compare the intensive modifications of distinct formalities can be thought of as at once building on many of Scotus's fundamental insights while also repairing the deficiencies in Scotus's account, filling out many of the sketchier details in his general approach, and providing a *geometry* to realize the vision of things only dimly captured by Scotus's more metaphysical discussions. We turn to Oresme now.

Oresme

Oresme begins his treatise *De configurationibus qualitatum et motuum* by appealing to the ancient distinction between geometry as dealing with magnitude (*megathos*) or the continuous, and arithmetic as dealing with multitudes (*plethos*) or the discrete. He then says:

> Every measurable thing except numbers is imagined in the manner of continuous quantity. Therefore, for the mensuration of such a thing, it is necessary that points, lines, and surfaces, or their properties, be imagined. For in them (i.e. the geometrical entities), as the Philosopher has it, measure or ratio is initially found, while in other things it is recognized by similarity as they are being referred by the intellect to them (i.e., to geometrical entities). Although indivisible points, or lines, are non-existent, still it is necessary to feign them mathematically for the measures of things and for the understanding of their ratios. Therefore, every intensity which can be acquired successively ought to be imagined by a straight line perpendicularly erected on some point of the space or subject of the intensible things, e.g., a quality. For whatever ratio is found to exist between intensity and intensity, in relating intensities of the same kind, a similar ratio is found to exist between line and line, and vice versa. For just as one line is commensurable to another line and incommensurable to still another, so similarly in regard to intensities certain ones are mutually commensurable and others incommensurable in any way because of their [property of] continuity. Therefore, the measure of intensities can be fittingly imagined as the measure of lines, since an intensity could be imagined as being infinitely decreased or infinitely increased in the same way as a line.[95]

[95]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, I.i.

We will see what it means both (i) for every intensity to be "imagined by a line" and (ii) for there to be a correspondence, in relating intensities "of the same kind," between the ratio of two intensities and a "similar ratio" of two lines. According to Oresme, when we speak of an intensity, in general we mean "that according to which something is said to be "more such and such," as "more white" or "more swift.""[96] Intensive alterations, such as change in the degree of heat of part of a body or the shade of a color, are to be distinguished from changes in extension, such as change in the *length* of a heated pipe. But why are such intensities represented with *lines*? Oresme observes that

> Since intensity, or rather the intensity of a point, is infinitely divisible in the manner of a continuum in only one way, therefore there is no more fitting way for it to be imagined than by that species of a continuum which is initially divisible and only in one way, namely by a line. And since the quantity or ratio of lines is better known and is more readily conceived by us—nay the line is in the first species of continua, therefore such intensity ought to be imagined by lines and most fittingly by those lines which are erected perpendicularly to the subject. The consideration of these lines naturally helps and leads to the knowledge of any intensity.[97]

In other words, because intensities are infinitely divisible in precisely the way the standard continuum (a line) is infinitely divisible, and because the ratio of lines and lines is readily conceived by our intellect, intensities are most fittingly imagined by lines, specifically by lines "erected perpendicularly" in relation to a subject of the intensible things, e.g., a quality (we will see what this looks like below). This allows Oresme to claim that equal intensities will be designated by equal lines, "a double intensity by a double line, and always in the same

[96] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 167.

[97] Ibid.

way if one proceeds proportionally."[98] And this is not to be restricted to certain qualities, such as sensible qualities, but is to be applied much more widely:

> This is to be understood universally in regard to every intensity that is divisible in the imagination, whether it be an active or non-active quality, a sensible or non-sensible subject, object, or medium. For example, it is to be understood in regard to the light of the body of the sun, to the illumination of a medium, or to a species in the medium, to a diffused influence or power, and similarly to others [...].[99]

Oresme proposes to realize this "consideration of lines" by representing the intensity as the "latitude" or "altitude" of a quality, which is to be paired with the "longitude" of the quality, regarded as the "extension of any extended quality" and represented "by a line drawn in the subject, a line on which the line of the intensity of the same quality is erected perpendicularly."[100] While there are many natural reasons for having expressed things the other way around—namely intensity by longitude and extension by latitude—the important thing is that there is a "dependence" between the two that can be captured by imagining the two lines to be perpendicular, where the *extension* of a quality is its longitude and *intensity* its latitude or altitude:[101]

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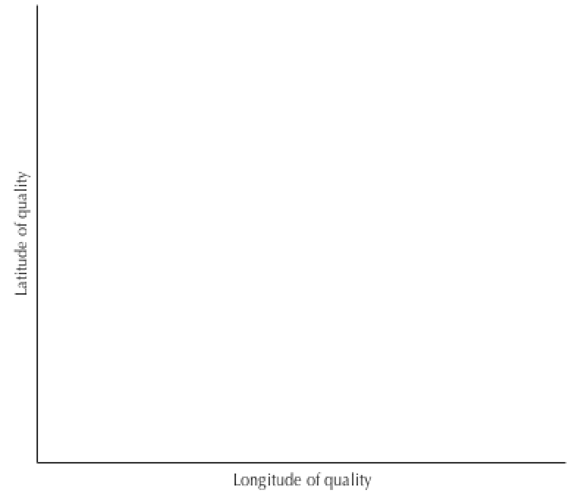
[98] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 167.

[99] Ibid., 169.

[100] Ibid., 171.

[101] Ibid.



Since both the longitudinal and latitudinal parameters can vary, what Oresme calls a "linear quality" or "the quality of some line in the subject informed with a quality" will give rise to "a surface whose length or base is a line protracted in a subject of this kind [. . .], and whose breath or altitude is designated by a line erected perpendicularly on the aforesaid base,"[102] where "the quality of a surface is imagined as a body whose base is the surface informed with the quality."[103] In this way, Oresme introduces his "figuration of qualities." We will look at some of these figures in some detail below. But first note that the only major restriction Oresme places on this figuration is the following: "no quality is to be imagined by a surface or figure having an angle at the base greater than a right angle; or by a segment of a circle that is greater than a semicircle," for such an intensity would "actually be outside of the subject."[104]

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[102]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 175.

[103]Ibid., 177.

[104]Ibid., 179.

In figuring qualities in this way, Oresme insists that it is never a question of particular fixed units, but rather always of ratios or proportions:

> a figure erected on a line informed with a quality is said to be "proportional in altitude to the quality in intensity" when any two lines perpendicularly erected on the quality line as a base and rising to the summit of the surface or figure have the same ratio in altitude to each other as do the intensities [of the quality] at the points on which they stand.[105]

As long as it is the case that, for any two distinct "figurations" (and for any two lines in the respective figurations), changes in altitude are represented in the same proportions to one another, it is "a matter of indifference" whether a quality is imagined by a greater or lesser altitude, surface or figure (one with a greater or lesser area), or with a dissimilar figure. Yet it does not follow from this insistence on the ratio of intensities that any quality can be imagined by any figure whatsoever:

> Indeed no linear quality is imagined or designated by any figure except the ones in which the ratio of the intensities at any points of that quality is as the ratio of the lines erected perpendicularly in those same points and terminating in the summit of the imagined figure [. . .]; from this it is apparent that a quality of this sort cannot be designated by a rectangle or by a semicircle; and similarly concerning an infinite number of other figures.[106]

In other words, there must be a certain *coaptatione*—"fittingness" or "suitability"—between the figure and the quality designated thereby. For instance, concerning the so-called "right-triangular quality," Oresme says:

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[105]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 181.

[106]Ibid.

> Every quality which is imaginable by a triangle having a right angle on the base can be imagined by every triangle having a right angle on the same base; and by no other figure can it be imagined. That some quality is imaginable by such a triangle is evident from the preceding chapter because some quality can be proportional in intensity to such a triangle in altitude. This quality is that which is commonly called a "uniformly difform quality terminated at no degree." However, more properly it can be called a quality uniformly unequal in intensity just as the triangle to which it is proportional is uniformly unequal in altitude. Similarly it would be better to say that it is terminated at the "privation" [of the quality] rather than at "no degree."[107]

There are also qualities imaginable by triangles that does not form a right angle with the base. But such figured qualities are in fact "divisible into two qualities, each of which is imaginable by a triangle having a right angle on the base."[108] Since no figure can have an obtuse angle on the base, every triangular quality will be "assimilated either to the triangle having a right angle on the base. . . or to a triangle having two acute angles on the base."[109] To what extent, then, does a quality determine the angle used in the figuration of that quality? For Oresme, "no quality, whatever its quantity, determines in itself the quantity of the angle used in its imaginable figuration,"[110] except for the following:

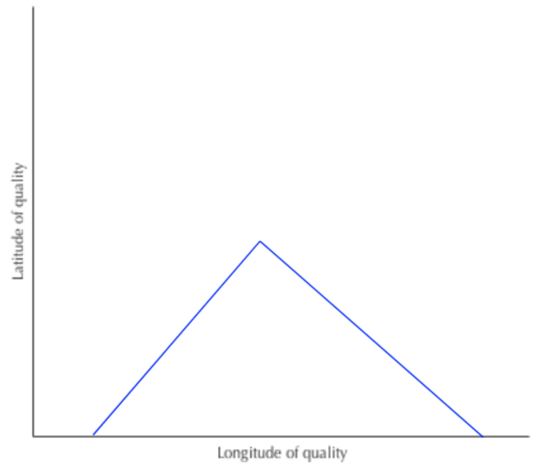
1. uniformly difform quality terminated at no degree:

[107]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 185.

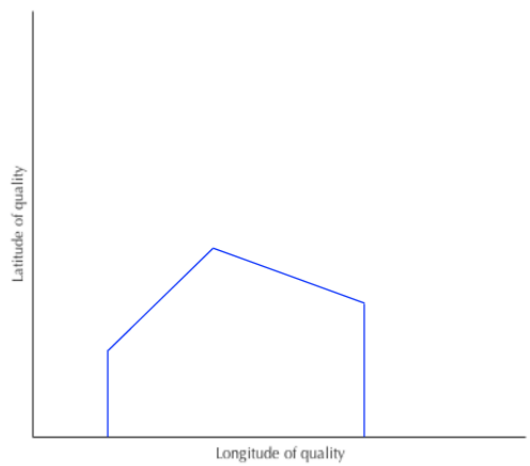
[108]Ibid., 187.

[109]Ibid.

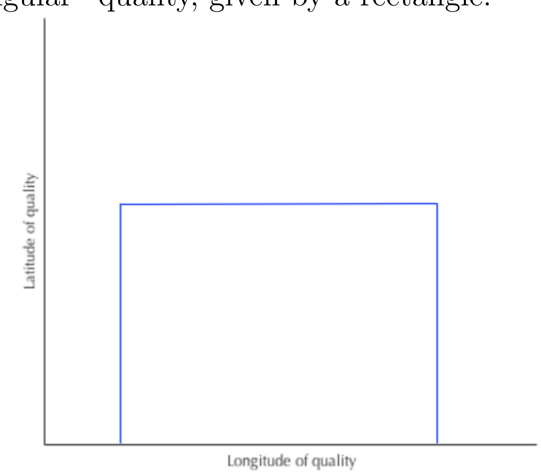
[110]See ibid., 191.



2. uniformly difform quality terminated in both extremes at some degree, having two right angles on the base:



3. uniform or "quadrangular" quality, given by a rectangle:



Every other quality, besides these three, is said to be "difformly difform" and is "imaginable by means of figures otherwise disposed according to manifold variation." A difformly difform quality can initially be described negatively as a quality that is "not equally intense in all parts of the subject nor in which, when any three points of it are taken, the ratio of the excess of the first over the second to the excess of the second over the third is equal to the ratio of their distances."[111] Put otherwise:

> if the line of intensity or summit line is a curve or is composed of several lines rather than one, then the quality imaginable by that figure will be difformly difform, and it can be that it is terminated in both extremes at some degree, or in both extremes at no degree, or at some degree in one extreme and at no degree in the other.[112]

Oresme holds there to be two types of difform difformity: simple and composite. Simple difform difformity is whatever can be designated by a figure whose line of intensity is not composed of several lines but is rather a single line, i.e., the line is in fact "necessarily a curve."[113] There are four kinds of simple difform difformity:

> (1) is imaginable by a figure which is not a segment of a circle nor proportional in altitude to some segment of a circle but whose summit is determined by an irrational curvature, or (2) is imaginable by a figure whose summit is determined by a rational curvature, namely, by a circular figure or one proportional to it in altitude. And each of these two kinds of figures can be either convex or concave. According to these four differences, then, there are four kinds of simple difform difformity, namely (1) rational convex, (2) rational concave, (3) irrational convex, and (4) irrational concave.[114]

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[111] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 195.

[112] Ibid., 199.

[113] Ibid.

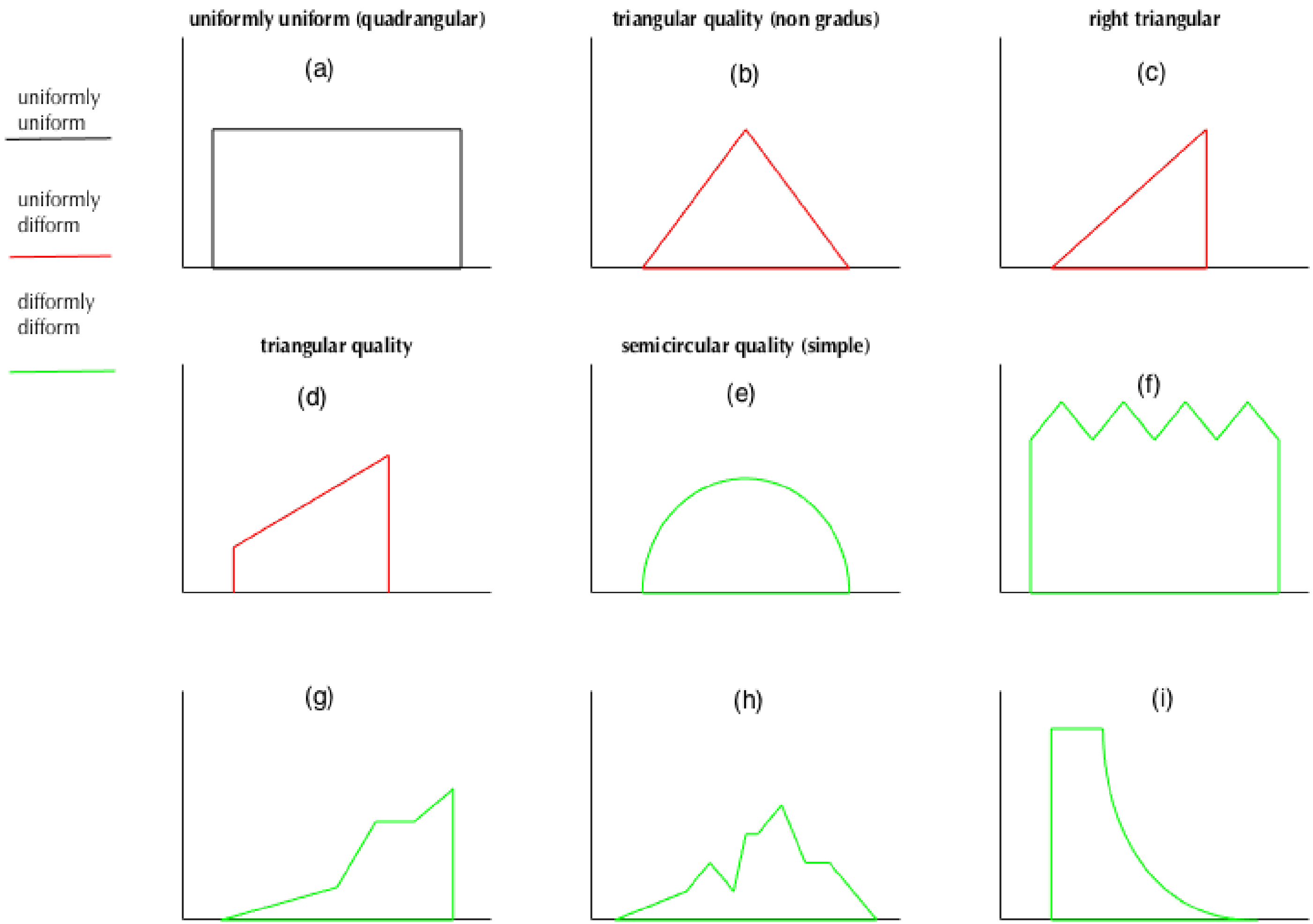
[114] Ibid., 203-5.

Oresme refers to these four main differences as "essential differences"; in contrast to these, there are "other accidental differences," differences that include whether the qualities are terminated at some degree or at no degree, i.e., having to do with the different combinations of altitudes at the extremities.

Beyond the four kinds of simple difformly difform quality, there is also "simple uniformity" and "simple uniform difformity," making six kinds of simple figuration of qualitative intensity. *Composite* difform difformity is made out of combinations of several of the six basic types of figuration, yielding sixty-three species of composite difformity in total.[115] Add to these sixty-three the four basic species of simple difform difformity for a total of sixty-seven kinds of difform difformity. Oresme observes that within these types, however,"there can be infinite variation by reason of the number and by reason of the order or disposition of the simple figures of which these species are composed."[116] In sum, then, Oresme produces qualities of three main types: uniformly uniform (line parallel to the line of longitude); uniformly difform (linear, with possibly one or two right lines to the longitude); and difformly difform (non-linear). By combining these types and composing new figures out of various "simpler" figured qualities (of the same or different type), a variety of figured qualities can be produced. I have constructed some examples of these below and arranged them in a table. As far as the difformly difform figures go, the table is obviously not meant to be complete, but only to give a more definite idea of the main figures:

[115] $\binom{6}{1} + \binom{6}{2} + \cdots + \binom{6}{6}$.

[116] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 207.



If this "figuration" were all that Oresme had accomplished, one might be impressed yet somewhat skeptical of its value. However, beyond just figuring a given form or quality in terms of its longitude and latitude, Oresme considers the *mensura* or quantity of the form: this is given by the area of the representative figure, something that would allow for the comparison of such figurations. It is also worth noting that Oresme does not stop at two-dimensional representations; his analysis is extended to three (and, in principle, even higher) dimensions:

> For just as certain linear qualities are uniform, others difform or uniformly difform or difformly difform in a multiplicity of ways, so one can correctly [speak] in the same way of surface qualities. And just as uniform linear quality is imaginable by a rectangle, so uniform surface quality is to be imagined by a body having eight three-dimensional right angles, and such a body can be imagined as being higher or lower with the quality remaining the same, just as was stated concerning linear quality.[117]

Where before he spoke of rectangles, he now speaks of surfaces; where he spoke of lines, he now speaks of planes; where he spoke of curves, he now speaks of curved surfaces. In fact, he claims that all that has been determined thus far is "just as appropriate" for linear qualities as it is for surfaces or "even a corporal quality."[118]

But while Oresme lays out a way of comparing such figurations of qualities, what does this comparison actually *mean* or indicate in terms of the underlying qualities? For Oresme, there is a close connection between the *powers* and *affections* of bodies or corporeal qualities and their figuration: "natural bodies, when mutually compared, according to configurations of this sort have mutually different operations and are differently affected."[119] Building on Scotus's notion that powers are rooted in intensive magnitudes, Oresme elaborates on this notion that "bodies have an efficacy or power arising from a natural figuration of active quality":[120]

[117] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 209.

[118] Ibid., 211.

[119] Ibid., 235.

[120] Ibid., 231.

> bodies can act in different ways as the result of a variation in the shapes of these bodies [. . .]. And since this is the case in regard to the shapes of bodies, it seems reasonable to speak in a comformable way concerning the previously described figurations of qualities. So, if there is a quality whose particles are proportional in intensity to small pyramids, it is accordingly more active, other things being equal, than an equal quality which is simply uniform, or which would be proportional to another figure not so penetrating. Or, if there were two qualities and the particles of one were proportional to more acute pyramids than the particles of the other, the quality corresponding to the more acute pyramids would be more active, other things being equal, and similarly for other figures. For it has become known by experience that a quality uniformly extended in a subject, e.g., hotness, acts differently, and alters the sense of touch differently, than does an equal intensity [whose particles of quality vary as follows:] one of its particles is increased in intensity, a second is decreased in intensity, a third increased, and so on alternately through the particles of the subject, so that this quality would be difform and, according to the imagery posited, would be figured by means of small pyramids. And so, according to this, what is commonly said is perhaps true, namely, that some qualities are pungent, e.g. a taste or an odor, or a cold or hot quality, like the hotness of pepper. And sometimes one finds two qualities of the same species which are equally intense and yet one is more active and more pungent than another. The cause of this can be assigned by following the imagery already described.[121]

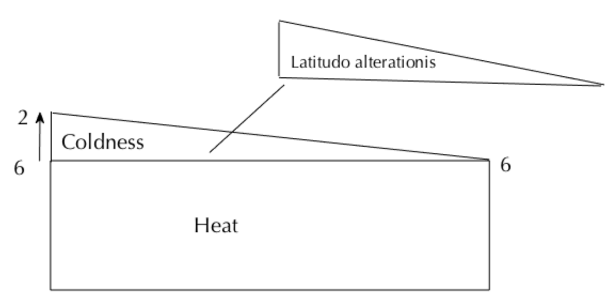
This is to realize geometrically and give shape to many ideas already implicit in Scotus's algebra of modes, and also to address problems raised by contemporaries such as Henry of Hesse (such as the fact that two qualities of the same species which are equally intense can be such that one is "more active" than another). It it no accident then, that in explaining his final remark from the passage above, Oresme also connects these ideas to issues surrounding Scotus's addition theory.[122] Oresme observes that in addition to the properties of the

[121]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 227-9.

[122]Another issue lingering from Scotus concerned how two "contraries" of a quality could be "mixed" or found simultaneously, an issue Oresme resolves by arguing that there is not "true contrariety" (*vera contrarietas*) in the ratios of figures. In terms of the figurations themselves, Oresme's general approach

figurations, another cause of this fact (that two qualities of the same type which are equally intense can be different in their "activeness") is to be found in the further fact that "few or no mixtures [i.e., composite bodies] are simply and thoroughly homogeneous; and even if they are of a single genus in substance, still they can be difform in quality."[123] Moreover, beyond the "shape" such qualities may receive from their subject, i.e., beyond the fact that "the form of a lion demands a different corporeal shape than does the form of an eagle," the qualities or forms "must be figured with a figuration that they possess from their intensity," and so "the natural heat of a lion is, in respect to intensity, figurable in a different way than is the heat of an eagle or a falcon; and similarly for others."[124] In this way, Oresme admits the possibility of a sort of determination of form coming from *the characteristic way that form or quality changes in intensity* rather than from the subject or substance. And the dissimilarities in these characteristic ways a quality changes in intensity will be reflected in

allows one to easily see how a single figure might in fact be composed of contraries, e.g., a given trapezoidal figure as decomposable into the following contraries:



[123]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 229.

[124]Ibid., 233.

the dissimilarities of the figurations of intensities, dissimilarities that further account for the "diversity of powers and actions":

> [T]he natural heat of a lion is active in a different way than is the natural heat of an ass or an ox, and it has a different power, not only because it is more or less intense, or has some such difference, but also because it is otherwise or dissimilarly figured in regard to intensity. It is the same for other qualities of these and other natural bodies.[125]

It is from precisely this analysis that Oresme derives a "universal rule": namely "that the measure or ratio of any two linear or surface qualities or velocities is as that of the figures by which they are comparatively and mutually imagined."[126] With these two ingredients—the figuration of a quality's intensity and the ability to compare two (or more) qualities by looking at the ratio of their figures—Oresme is able to breath geometrical life into many of Scotus's more metaphysical ideas and give rise to a more nuanced and tractable account of the sorts of continuous variations supported by intensified qualities or forms.

Oresme will go on to speak of the "perfection" and "excellence" of certain qualities and their figurations, a discussion that takes a number of other Scotistic ideas in new directions, as well as leading to some of his most transformative ideas regarding continuity. These ideas

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[125] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 235.

[126] See ibid., III.v, 405. At first glance, this may seem like a rather speculative and even idle claim; but it is in fact on the basis of this rule, found in his long discussion of measuring difform qualities, that Oresme comes to argue that "Every quality, if it is uniformly difform, is of the same quantity as would be the quality of the same or equal subject that is uniform according to the degree of the middle point of the same subject" (409). And this is nothing other than the well-known Merton College Rule of uniformly difform motion—or the "Mean Speed Theorem"—for which Oresme is perhaps most famous today and that arguably anticipated and prepared later results in physics, especially Galileo's development of kinetic laws.

appear to be influenced, more or less explicitly, by reflections on musical matters.[127] Oresme begins by observing how

> just as it has been demonstrated in the theory of music that certain ratios are more perfect and delightful than others not only in sounds but in other things as well[...], so also is it certain that some corporeal figures excel others in beauty and are simply [*simplicter*] nobler and more perfect. [...] Therefore, in the same way it seems reasonable, in regard to the difference in configurations of qualities posited above, that those qualitative configurations which are similar and proportional to nobler and more beautiful or more perfect corporeal figures are simply better or nobler.[128]

Elaborating on this idea that there are differences in perfection that stem from differences in the configurations of qualities and their proportions, Oresme conjectures that "it seems to follow that those species which in themselves determine such nobler configurations of qualities are of a nobler constitution and a more perfect nature."[129] Not just that, but even within the same species, "the one individual is assumed to be better constituted [*melius complex-*

[127]Unfortunately, I do not have space to elaborate much on the little-explored line of influence going from seemingly technical investigations in musical theory and notation (both those authored by Oresme and by others of his time) and Oresme's entire project concerning the quantification of intensive changes, his configurations, as well as his more mathematical treatment of the commensurable-incommensurable. Of course, *ars nova*, with its burgeoning development of counterpoint and polyphony, is in the background. But more generally, musical notation at the time represented sounds in terms of *extensio* (time) and *intensio* (pitch), a practice that may have had some influence on Oresme's development of figurations. There were also many issues involving mensural notation (i.e., issues in rhythm) and technical matters involving *tempus* and *modus*, as well as questions concerning the limits of division and finding proofs for the exhaustiveness of permutations of certain note values, all of which appear to be connected with issues of "perfection" for those who dealt with these matters. For instance, in Muris's *Notitia artis musicae* (which text the editor suggests Oresme probably knew; see Oresme, *Algorismus proportionum, 1358 De Proportionibus Proportionum*), Muris attempts to reconcile the standard idea that perfection is to be found in ternary values with the practically useful 'imperfect' binary or duple divisions of note values. His discussion explicitly engages more philosophical disputes over perfection and imperfection. Musical matters likewise seem to be in the forefront of Oresme's mind throughout his development of his configuration theory; we will see perhaps the most conspicuous instance of this in what follows.

[128]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 239.

[129]Ibid.

ionatum], other things being equal, whose fundamental quality possesses more fittingly the mode of configuration due to its species."[130]

> Thus for nobility of constitution [*nobilitatem complexionis*] there is required not only a nobler ratio of intensity or remissness of primary qualities but in addition a noble configuration of these same [qualities] or of another quality. This is because each of these differences, namely in the ratio of the qualities and in their configurations, makes for a difference in constitutions, [both] in diverse species and in the same species. For it is perhaps possible that there is the same or a similar ratio of primary qualities in each of two individuals that differ in species, but that these individuals differ in species and perfection because of the diversity of the configuration of their constitutional qualities. And similarly in the same species [they differ] according as one participates more perfectly in, or attains more closely to, the most perfect configuration of qualities due its species, or also conversely. For example, it is possible that a horse and an ass, or two horses, are in accord insofar as the figuration of qualities is concerned but that they differ in the ratio of these same qualities, or also conversely. Nor is it necessary that the configuration of qualities which is most perfect or most beautiful for this species is simply [*simpliciter*] more perfect. But it suffices only that it is the most fitting and most beautiful configuration for that species.[131]

This idea that the ratio of intensities of different qualities contributes to differences among beings and to their perfection, as well as the idea that a configuration of qualities is not more perfect *simply*, but rather to the extent that it is the "most fitting configuration" for a given type of species, are ideas that will get developed by Spinoza to great effect, as we will see in the subsequent chapter. Oresme pushes the musical connection even further in the consideration of the comparisons of ratios of qualities, developing the idea of *consonance* (or *comformity*), and *discord*, between the ratios of qualities:

[130] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 241.

[131] Ibid.

> still it is certain that some ratios seem naturally to accord [*convenire*] better with each other, just as a rational ratio accords better with a rational ratio than it does with an irrational or surd, and a harmonic ratio better with another harmonic than with an non-harmonic [*enarmonica*] ratio. So it is also, in regard to figures, that certain of them are more conformable to, and consonant with [*magis conformes et magis consone*], each other than to others. Thus when one [of such conformable figures] is inscribed in, or circumscribed about, the other, or compared to it in some other way, it is related to its comformable figure more beautifully than it is to some other figure. As for example, perhaps a square is more beautifully related to a circle or an octagon than it is to a pentagon. The same thing is true regarding the aforesaid configurations of qualities: some are mutually conformable or fit together better while others do not fit together well.[132]

Oresme draws all of this together, claiming that this account of the mutual comformability (or lack thereof) of various figurations of intensities is even what ultimately accounts for the differing relations between diverse species. Together, the two causes—namely "the relation of the ratios of natural qualities and the relation of their configurations"—join together "to produce either the natural friendship or natural hostility of one species toward another [*ad amicitiam vel ad inimicitiam naturalem*]."[133] In other words, the principal cause of the "natural friendship between man and dog" is to be found in "the fitting accord between the configurations of the primary or other natural qualities in each of these species."[134] In a very important passage, Oresme clarifies that he is here speaking "of the fitting accord [*convenientia*]—not that of closeness, but that of conformity [*non propinquitatis sed confor-*

[132]Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 241-243.

[133]Ibid., 243.

[134]Ibid.

mitatis]."[135] In elaborating on this distinction between conformity and closeness, Oresme again appeals to music, and here we begin to appreciate the true force of this appeal:

> Just as we see in music that it is not by the degree that one sound is closer to another that it is more harmonious with it but rather that a fitting ratio is required, so here closeness is not the attendant measure but rather a fitting and natural conformity.[136]

On the continuum of sound, the closest notes—beyond the "just noticeable difference," where the two are first detectable as non-identical—will usually be *the most dissonant* or the "furthest" from one another in terms of any structural conformity between the intensities. On the other hand, the mutual agreements and more composite unities produced by certain combinations of notes, are forged along the lines of some structural conformity between the ratios of intensities characteristic of each separate sound—something that has nothing to do with closeness in the sense of proximity. Those sounds that combine well with other sounds do so on account of the greater conformity between their ratios of intensity of that quality, and *not* on account of any uniform relationship of *closeness* imposed on the continuum of the quality (sound) from without. In speaking of intensities figured with continuous geometrical figures, and in developing the relations between these in terms of ratios of such intensive magnitudes, Oresme extends these features of the musical continuum to continua more generally, allowing him to advance a concept of continuity freed from closeness and uniform distance metrics.

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[135] Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 243.

[136] Ibid.

On this account, discord (*disconvenientia*) is explained in the obvious and contrasting way, and is even what is chief among the causes of "natural hostility" to be found in nature between different species and different beings.[137] Likewise, this account is meant to help explain the differences in affections between different beings, and so to account for such things as the fact that

> thorns [unpleasant for humans] please the ass, while pottage [tasty to humans] displeases the ass. Also, something pleases a well man and is displeasing to a sick one, and there are many such examples. It is the same for something that can be heard. For certain animals like music and others do not, and men differ much in this regard.[138]

Oresme points out that while "many causes may concur in producing this diversity, still it perhaps seems probable that the two aforesaid [causes] are the most important," namely (1) the accord or discord between the ratios of intense qualities, and (2) the accord or discord between their configurations.[139]

In summary, then, Oresme has offered a theory of the "natural friendships" and "natural hostilities" between beings, and of the differences in affections, based ultimately on the "figuration" of intensities "in the manner of continuous quantity" via lines and surfaces. Such intensities are represented in this way, with lines, precisely because intensities are "divisible in the manner of a continuum." But because every intensity—from "active or non-active quality" to a "sensible or non-sensible subject, object, or medium"—can be thus represented, this allows Oresme to lift the geometrical treatment of the continuum, and

[137] See Oresme, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, 243.

[138] Ibid., 249.

[139] Ibid.

of properties of continuity characteristic of geometrical objects and magnitudes, into the treatment of all kinds of qualities and forms. Different powers of bodies are then even explained in terms of dissimilarities in the figuration of the relevant intensities. Moreover, relations between one intensity and another are held to be as ratios of magnitudes. How well one figuration of intensity combines with another gives rise to "agreements" that are ultimately responsible for the harmonies and disharmonies found between different beings throughout nature.

The most important insight running through this entire theory is its transformation of the concept of *continuity*. With Oresme's theory, *continuity* can now at once be treated as something that is an *intrinsic* property of certain things (like forms or qualities) and, in passing to the *relations* between various intensities of qualities, as something that is preserved not through closeness, but through *structural or morphological conformity between ratios of those intensities*. I mentioned at the beginning of this chapter that this detachment or "freeing" of continuity from *closeness*—in the form of proximity or similarity, both of which ultimately force continuity to be a matter of purely *extrinsic* determinations and which gives rise to a monolithic notion of measure in the form of a reduction of all relations between degrees to one of "distance"—in the name of conformity between the configurations of intensive qualities and their ratios is perhaps the single greatest and most influential transformation to the concept of continuity. In addition to clearly influencing Oresme's thinking on these matters, music offers a powerful window into this transformation. Consider how, on the continuum of sound—say, for concreteness, on that portion that is audible to humans, from roughly 20 hZ to 20,000 hZ—arbitrary selections of notes, when related to one another and combined, will in general not yield mutually compatible or consonant sounds.

Accord or consonance is a relatively special and rare relation. Moreover, those notes that do combine well together do so on account of having the greatest conformity between their ratios of intensity; these are decidedly *not* the notes that are *closest* in the sense of the most proximal, as the closest notes in this latter sense will typically be *the most dissonant* or the "furthest" from one another in the sense of being highly incompatible or having highly non-conformable or "disagreeable" characteristic ratios. The unities produced by the combination of distinct notes, then, proceed along the lines of the morphological alignment of certain intensities and their ratios, not along lines of closeness. Here, in the continuum of sound, we glimpse a property that may apply more widely to other natures regarded as continua: that continuity is not always or even primarily a matter of closeness, but is to be found in morphological agreements or structural conformities that obtain between distinct intensities and their distinct manners of variation. The power of this idea is due, in large part, to Oresme's bold attempt to "figure" all kinds of intensities and qualities, so that the "accords" between magnitudes can be extended to an analysis of nature more broadly—ultimately leading to a theory of nature as being differentiated along lines of "natural friendship" and "natural hostility," itself unpacked in terms of agreements and disagreements between the ways the forms and qualities that different beings support change in intensity. It would be difficult to overstate the significance of this idea. The impact of this move on the future conceptualization of continuity (as well as the sorts of things held to be continua), moreover, is something that, while initially operating rather quietly and in a way that is easy to overlook, only becomes more and more *intense* (pun intended) as history progresses.

In the next chapter, we look at how Spinoza would revisit and reinvigorate many of these ideas and advances of Oresme, admittedly sometimes with different angles of approach and with different aims, but always in a way that seems to exhibit a remarkable “sympathy” with the fundamental ideas of Oresme. As far as the concept of continuity is concerned: I understand Spinoza to play an absolutely pivotal role in attempting to reconcile the tradition of continuity-as-closeness (that began with Aristotle) with the Oresmian model of continuity-as-conformity-of-ratios. Much of Spinoza reads like an elaboration on Oresme’s new model of continuity; however, the legacy of the Cartesian emphasis on local motion makes it difficult for Spinoza to give up on closeness altogether, and so in many ways he must strive to re-unite aspects of these two models.

Chapter 4

Spinoza's Physics and the Shadow of Descartes

Spinoza's Physics

Introduction

Everyone, including philosophers of all stripes, speaks of *individuals*. What distinguishes some, however, is a commitment to the non-scalability of the concept of an individual. Others admit the scalability of the notion of individuals—they are willing to commit themselves to the claim that it is more than a mere matter of words or "convenience" or perspective that an oxygen molecule is an individual that can be part of an individual red blood cell that can be part of an individual blood circulatory system that can be part of an individual human body that can be part of. . . . In other words, they hold to the belief that there are no *a priori* restrictions on the "level" of compositeness at which the concept of individual can be meaningfully applied. The first group, on the other hand, restricts *on principle* the applicability of the concept of individual to that which is least composite—or, assuming such things exist, to what is "simple" or non-composite—beyond which our license to apply the concept can only be a matter of "convenience."

While this is not a chapter on the virtues or demerits of these two positions (if I can call them that), I can remark that the restrictions of the first group are, on their own terms, *very* difficult to uphold consistently in practice. However, the real merit of such a position lies in the challenge they present to the second group: "you say that the concept of individual can be scaled, but prove to me that this scalability of individuality reflects a fundamental feature of nature and is not just a matter of a certain flexibility in the use of words!" This is a very important and serious challenge, one that could only be met with a strong theory capable of accounting for what makes something a ‘one’ and for how such individuals can be *ordered* (e.g., one individual subordinated to another as a part, while remaining one in its own regard). Such a theory would amount to a natural theory of *composition*.

In attending to any number of examples of the sorts of things that are fairly uncontroversially regarded as individuals or ‘one’, one can observe that the fact that something has (or has not) parts is not in fact what is decisive to its being ‘one’. Until the first group above proves otherwise, there seems no reason to take their negative hypothesis to be true, namely that individuality or ‘oneness’ has a special relationship with *simplicity* or the lack of parts. So we might instead begin with the assumption that an individual, whatever else it is, has (or at least may have) parts. But in reflecting on our many examples, we observe that there does not appear to be anything in the parts themselves that make our individual what it is: the same parts can be combined in various ways to different effect, just as different parts can be combined in various ways to produce the same effect. So we begin to suspect that, perhaps, individuality is less about the component parts and more about the relations between those component parts, their manner of combining. But, as we explore this idea, we are again struck by the consideration of a number of examples that lead us to admit that we do not seem to require that the parts retain the same configuration, that the same parts always participate in the configuration, or even relate to one another always in exactly the same manner. Rather, it seems that the constitutive, “individual-making” relations between the parts are not static or uniform. Yet we of course require that whatever is “individual-making” is invariant, or at least robust to certain changes. How do we proceed?

According to Spinoza, the individuality or 'oneness' of each thing is constituted by a certain invariant manner or pattern according to which a number of bodies *communicate their motions* among themselves, a constitution that equally contributes to that body's manner of communicating the motions of certain *external* bodies—both how it propagates those motions throughout its parts and how it communicates the concerted motions of all its parts together to external bodies. Each is its own body by virtue of the unique way it circulates motions among its parts, "processes" the motions of other bodies and passes on or broadcasts its own motion "as a whole" to other bodies. When one body *does* the same thing as another, or concurs together in acting so as to produce a single effect, we are tempted to say that these bodies are *one*; but we are tempted thus because, in that moment, there is a temporary (and possibly only partial) alignment between how each of the bodies have had their component parts arranged in relation to one another so as to relay information, specifically motions, in a certain way. In other words, succumbing to such a temptation is only valid to the extent that we realize that such an account—that bodies are one either when they concur together in acting, or *do* the same thing—is at best a special case of the more general communication account, on which the former is derivative.[1]

[1] Spinoza commentators—and even the present author, in less careful moments—are wont to speak of the "power to act" of a body as what makes it the body it is, i.e., as vital to make a body the 'one' or unity that it is. The problem with this is not that it is wrong per se, but that it is merely derivative in relation to his more fundamental theory of the communication of the motions of its parts as what make a body *one*. Bodies that "concur in acting" to produce one effect can be called a "singular body," it is true; but it is not on account of such "concurrence in acting" that there is an *individual* or *one body*. Moreover, I can use my hands to hold my coffee but that does not make my hands the same individual or body as a cup. And the point is—why not? The point is that it only really begins to make sense to say that a body *is* its power to act if we mean to include *everything* it could ever do or have done to it. But note that this would involve consideration of all sorts of extrinsic relations to external bodies, and is not something that could ever *explain* what made an individual that individual, and do so intrinsically. On the other hand, what a body can do or have done to it is something that follows from the particular fixed manner with which its component bodies or parts communicate their motions among themselves, i.e., "intrinsically." The latter can indeed explain the former, and can provide an *intrinsic* account of a body as one body, whereas the former

It may seem surprising, at least on the face of it, that someone such as Spinoza who argued that everything is ultimately included in one unique whole—a "monist"—could be responsible for a sophisticated theory of individuals or what makes a composite body 'one'. To paraphrase Hegel, monism would seem doomed to failure on at least one of two fronts: in not being able to account for how the determinations supplying the unity of each individual body could arise from the nature of the whole itself, thus relegating such distinctions to the status of being "alien to" the underlying whole in which they all participated or to the status of being mere distinctions of reason, conveniences of the human mind; or in consisting of nothing more than an abstract or "sham" unity of entities that are fundamentally and irreconcilably distinct. Yet, properly understood, Spinoza's powerful theory of individuals, or the unity of composite bodies, provides just such a sophisticated theory, without falling prey to the criticisms of the likes of Hegel.

Spinoza's "monism" of course requires that different things be different not as different *substances*, but in some other way. However, this way of considering things is, in a sense, to approach the problem from the wrong side. Spinoza's "monism" can more naturally and productively be seen as a *consequence* of his theory of how composite bodies are 'one' and how the notion of *one body* scales. His theory of the unity or 'oneness' of bodies can be found

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will amount to an entirely extrinsic account and will end up having to appeal to some "fact of the matter" (which, importantly, will never actually be available to any finite mind, for there will be no way of inspecting all a body's ways of acting on other bodies). And this is all as it should be, for there are serious limitations to any purely functionalist account of the 'oneness' of a body, which fails to acknowledge the fact that many things that would be uncontroversially distinct individuals can be made to do the same things and many things that can do the same things are not, for all that, ever regarded as identical individuals. On the other hand, we can easily envision understanding the fixed manner in which the parts of a body communicate among themselves, and can also understand how such an invariant structure would suffice to enable us to predict when two seemingly distinct things would in fact be capable of doing and having done to them *some or all of the same things*.

in the so-called "Physical Interlude" from Part 2 of the *Ethics*. This brief but decisive section can be effectively partitioned into two sub-sections, the first of which offers a terse account of the "simplest bodies" (*corpora simplicissima*) on which the second part, which constructs the outlines of a more sophisticated and impactful theory of how composite bodies achieve their unity or 'oneness', would appear to be based. The second section, on composite bodies, is centered around Spinoza's rich definition of what makes a body a 'one' or individual, after which many conditions under which such an individual can support certain changes without its 'oneness' being thereby undermined are spelled out. The pivotal definition reads:

> Definition: When a number of bodies, whether of the same or of different size, are so constrained by other bodies that they lie upon one another, or [*vel*] if they so move, whether with the same degree or different degrees of speed, that they communicate their motions to each other in a certain fixed proportion [*motus suos invicem certa quadam ratione communicent*], we shall say that those bodies are united [*unita*] with one another and that they all together [*omnia simul*] compose one body or individual [*unum corpus, sive Individuum componere*], which is distinguished from the others by this union of bodies.[2]

This definition has a number of decisive moving parts. The most important of these by far is the concept of *communication*. A number of bodies form *one body* (or an Individual) by virtue of how *the component bodies communicate their motions to each other in a certain fixed manner*. That which is invariant in such a pattern of communication of motions between parts—specifically those that move—is what makes for the stable and characteristic 'oneness' of any individual body. It is something of a scandal that nearly all commentators attribute to Spinoza—if not whole-heartedly, then at least in certain careless moments—the idea that a body is its "fixed ratio of motion and rest," something that gets bandied about almost as

[2] *Ethics*, II, Physical Interlude, Def; Spinoza, *The Collected Works of Spinoza, Volume I*, 460.

a mantra. This compact phrase has surrounded the working of Spinoza's theory of bodies and their composition in a kind of haze, giving one the artificial sensation that there is fundamentally something ill-conceived or deeply unclear in the account. To cut through the haze it suffices to attend to what Spinoza actually says, which we will do in this chapter and the next. For now, suffice it to say that Spinoza is always careful to phrase things in terms of the *communication* of motions between parts in a certain fixed ratio/pattern/way. This is *very* different from saying that a body is its "fixed ratio of motion and rest"—and that remains the case *however one choses to understand the term 'ratio'*. The tendency to attribute to Spinoza the idea that a body is its fixed ratio of motion and rest is so deeply entrenched, that it will be difficult for the experienced reader of Spinoza not to balk at the claim that Spinoza does not actually say this. But I believe it is rather easy to make a compelling case that this is indeed the case, and that the account of what makes a body an individual that we ought actually to attribute to him—his "communication" theory—is a much better, more coherent, and more far-reaching account anyways. The general failure to acknowledge this distinction, I believe, is largely responsible for the fact that there has been relatively little appreciation among commentators that Spinoza even *has* a coherent theory of composite bodies, let alone that it is a powerful and central part of much of his thought.

Another pivotal moving part lies in the seemingly innocuous word "simul," a word that will be shown to be doing a lot of work (especially in the context of Descartes' physics, which has a number of deep connections to Spinoza's physics, as we shall see). The word "simul," in looking back to Descartes's own definition of 'one body' packed within his definition of local motion, is something of a cipher inviting us to revisit and clarify Spinoza's relationship to Cartesian physics. A final key ingredient in this definition comes in the requirement, regarding a number of bodies not necessarily moving (and presumably distinguished from one another entirely in terms of a definite state of motion or rest), that they be *contiguous* or *close*. I will discuss this definition in far more detail throughout the next two chapters, but for now let me note that I take this "closeness requirement" (everything that precedes the "or") to apply first of all to bodies that, like the *corpora simplicissima*, are only characterized in terms of their definite state of motion or rest. As such, it applies to bodies that either do not move, or, to the extent that they move, this is all one can say about them. However, I do not understand the "or [*vel*]" in the definition to be *exclusive*, since it would not make sense to hold that a number of bodies in motion, communicating their motions to one another, are necessarily *not* constrained to be close, if that means they are precisely "far." So while the "closeness" requirement pertains first of all to bodies that are only characterized in terms of their definite state of motion or rest, it is not that the "closeness" requirement somehow *ceases to apply* to composite bodies communicating their motions among one another. Rather, because a number of moving bodies may themselves include many component bodies (themselves perhaps composed of further bodies), the patterns of communication that emerge between such complex bodies will involve, depending precisely on the extent of the compositeness of the body and participation of many bodies (possibly spread "far apart"), a gradual "relaxing" of this requirement. This means that it will in fact turn out that the "closeness" constraint will gradually be relaxed more and more (but is never abandoned entirely). But at least initially, it is significant that the parts (or "number of bodies") of an individual or 'one body' are not just any old parts, distributed arbitrarily in relation to one another, but are "constrained to lie upon one another." For all its simplicity, this seemingly innocuous modifier turns out to be a key component of the underlying understanding of motion and rest, of the definition, and of the subsequent theory built upon the definition. In brief, this closeness hypothesis ensures, even as it is gradually "relaxed" for a number of bodies in motion, that the interactions between bodies in states of motion and rest and the communication of effects are *localized.* And this is precisely what ensures that the notion of a "cause" even makes sense. For without such an assumption—in a universe where entities that were "far" could have just as much effect on one another as those that were "near," where everything could in principle depend equally on everything else, where some aspects of entities could not be changed without this affecting all the rest—one could not even get the notion of causality off the ground. Likewise, without such an assumption, it is arguable that the notion of "one body" or an individual would also fail to stick: for, as we will see, to discover the invariance in a body's manner of communicating motions is to discover that certain aspects are robust to certain changes, i.e., that certain properties stay the same while others change (and this could not happen in a universe in which everything depends equally on everything else, i.e., a universe without "causes").[3]

It would be difficult to overestimate the impact of this definition, and the reach of the theory of composite bodies built upon it. For, in building this theory in the "Physical Interlude," Spinoza constructs, step by step, bodies that are composites of composites of composites of. . . , all of which culminates in the radical claim, with which the interlude ends:

[3]If you are not convinced that a universe in which entities that were "far" had (in principle) just as much effect on one another as those that were "near"—in which there was no alignment between causality and closeness, where this means that, on the whole, things far apart have much less effect on one another than things close together—is also a universe in which each thing will (in principle) depend equally on every other thing, you should try to produce a constraint (that does not end up assuming that "far things have less effect") that could ensure, in principle, that not everything depends equally on everything else.

> And if we proceed in this way to infinity [i.e., conceiving composites of composites of composites of . . .], we shall easily conceive that *the whole of nature is one Individual, whose parts, that is, all bodies, vary in infinite ways, without any change of the whole Individual.*[4]

Since all existing modes are composite—in fact, they may even be infinitely composite—it is decisive to understand this theory. The particularly economical and (as will be seen) recursive theory of composite individuals is vital to understanding some of Spinoza's most fundamental claims, and can help provide a number of natural solutions to otherwise intractable problems in the interpretation of some of Spinoza's thought.

A close look at Spinoza's theory of what makes bodies 'one' and how the notion of 'one body' can be scaled—all the way up to Nature as a whole—will make a number of contributions to this dissertation's overall concern with the concept of continuity and its connections to generality. To the extent that commentators have acknowledged that Spinoza was concerned with, and had interesting things to say about, the nature of *continuity*, the focus is always on what he says about the continuum, the (in)divisibility of extension, and the nature of infinity. While these discussions are certainly relevant to his thoughts on continuity, I believe that his true theory of continuity, wherein his greatest contributions to the understanding and advancement of this concept are to be found, lies elsewhere, buried deep within his "physics," specifically as developed in his rather involved theory of composite bodies and what makes a body 'one' and one that is robust to all sorts of changes. It is here, also, in the development of the unity of composite bodies, that one finds not just his most sophisticated thoughts on the nature of part-whole but also the seeds of a number of very

[4] *Ethics*, II, Physical Interlude, L7S; Spinoza, *The Collected Works of Spinoza, Volume I*, 462, my emphasis.

far-reaching connections between continuity and *generality* (through the *common notions*). With the concept of *common notions*, the "foundations of our reasoning" (II40Schol.), we have our first access to adequate ideas. This concept is grounded in a fact discussed at the outset of the "Physical Interlude," namely that all bodies agree in certain things, as well as the fact that if something is common to our body and certain external bodies by which our body is affected, and is "equally in the part and in the whole of each of them," it will also be necessarily adequate in the mind.[5] A decisive corollary of this proposition is that "From this it follows that the mind is more capable of perceiving many things adequately as its body has many things in common with other bodies."[6] This key proposition, together with a number of other propositions, strongly suggests that, in coming to understand adequately, at the level of body this amounts to striving to compose with *more and more of Nature*. On many occasions, Spinoza is keen to draw our attention to the "easier" or "softer" point that "nothing is more excellent than those [natures] which agree entirely with our nature," since if, for example, "two individuals of entirely the same nature are joined to one another, they compose an individual twice as powerful as each one,"[7] from which

> Human beings, I say, can wish for nothing more helpful to the preservation of their being than that all should so agree in all things that the minds and bodies of all would compose, as it were, one mind and one body; that all should strive together [*omnes simul*], as far as they can [*quantum possunt*], to preserve their being; and that all together

[5]Spinoza, *Ethics*, II39.

[6]Spinoza, *Ethics*, II, 39 Cor.

[7]Spinoza, *Ethics*, IV 18 Schol.

> [*omnesque simul*] should seek for themselves the common advantage of all.[8]

However, over-emphasis on such a point can mask the "stronger" point: that since "a composite individual can be affected in many ways and still preserve its nature"[9] and since composite individuals can themselves include many composite individual bodies "of different natures," while it is true that an entity should never "from the necessity of its own nature, strive not to exist, or be changed into another form,"[10] reason's discovery of the common notions really amounts to a discovery of how to compose with *as much of Nature as possible*—which means even with individual bodies of another nature—while simultaneously preserving its own nature.

A reconstruction of Spinoza's most decisive thoughts on continuity will thus come from a thorough understanding of the subtle relations between (i) the nature of what makes a composite body *one* or an *individual*, and (ii) how the *composition* of bodies works, i.e., how the notion of 'one body' gets scaled. Further connections between continuity and generality will be established by consideration of the workings of the communities of composition that, at the level of body, the *common notions* represent. *Oneness/unity* of a body, *composition* of bodies into ever more composite 'ones', and *common notions*: in understanding these three ideas and their relations, we will construct a picture of Spinoza's profound contributions to the continuity-generality connection.

[8]Ibid.

[9]Spinoza, *Ethics*, II, Physical Interlude, L7 Schol.

[10]Spinoza, *Ethics*, IV 20 Schol.

The rest of this chapter and the following proceed as follows. The remainder of this chapter is dedicated to the following: (i) contextualizing and motivating the "Physical Interlude" from the *Ethics*; (ii) presenting Spinoza's account of the "simplest bodies" from the first half of the interlude; (iii) showing how this is best situated in terms of aspects of Descartes' account of bodies in states of motion and rest from the *Principles of Philosophy*, in the course of which much of the Cartesian theory will be reconstructed; (iv) demonstrating that the central Cartesian definition of *local motion* and "one body" is not, as most commentators suggest, "viciously circular," but rather deliberately recursive, a feature that is in fact productive of a kind of deep coherence between the account of the simplest bodies in motion (an account that appears to be largely lifted from Descartes) and the Spinozist theory of composite bodies that comes to be built upon it; and (v) establishing some of the subtle but important structural connections between the Cartesian definition of local motion and Spinoza's theory of composite bodies. On the rare occasions that the originality and power of Spinoza's theory of the unity of composite bodies *is* acknowledged, this is typically paired with a withering assessment of the allegedly mostly absent theory of "simplest bodies" on which it would appear to be based. And to the extent that it is acknowledged that the account of the simplest bodies can be very naturally understood as taken over from Descartes, there is then typically some reference to how the underlying Cartesian theory is "circular," defining bodies in terms of motions and local motions in terms of bodies. The implication is then that, in defining bodies in terms of motions and rests, but in implicitly importing the Cartesian account and allowing it to take care of itself, Spinoza's ultimate theory of bodies must fall prey to the same accusation of circularity. The problem with this tradition of interpretation is not just that there are little to no *arguments* that the

Cartesian account is “circular”; far more seriously, the issue is that, understood on its own terms, the account does not seem problematically circular at all—but is, productively and deliberately, *recursive*. I believe that the major part of the originality and power of Spinoza’s physics, as developed in the *Ethics*, stems from how consistently and extensively he develops the implications of this definition of local motion from Descartes. Moreover, in passing to “higher-order” interactions between composite bodies, the physics found in the *Ethics* builds on this general picture of local motion, taking it much further than Descartes himself was prepared to do, but setting out from many of the same assumptions and ideas. The present chapter is thus largely devoted to an extensive discussion of the relevant aspects of Cartesian physics.

The chapter to follow this one, Chapter 5, will (i) build on the work of the present chapter to provide a reconstruction of Spinoza’s theory of composite bodies and his communication theory of the ‘oneness’ of bodies; (ii) establish a number of natural connections between the reconstructed account and the concept of *common notions*, developing his original theory of *generality*; and (iii) indicate how this reconstructed account and overall interpretation of the connections between composition of bodies and common notions has the capacity to provide a clear resolution to a number of broader interpretive issues in Spinozism.

The Physical Interlude’s Theory of Corpora Simplissima

The “Physical Interlude” of Book II of the *Ethics* is wedged between a proposition concerning how the object of the idea constituting the human mind is the body and nothing else (IIP13) and a proposition concerning how the human mind is more capable of perceiving many things the more its body can be disposed in a great many ways (IIP14). One of the main purposes of this interlude is evidently to provide a foundation for the postulates concerning the "highly composite" human body in particular—an individual that is "composed of a great many individuals of different natures, each of which is highly composite"[11]—and to explain the nature of this composition more generally. Spinoza makes it clear in the Scholium of Proposition 13 that the claims made about the human mind and its relation to body are in fact "completely general and do not pertain more to humans than to other individuals, all of which, though in different degrees, are nevertheless animate," from which it further follows that "whatever we have said of the idea of the human body must also be said of the idea of any thing." Nevertheless, Spinoza goes on to point out, it cannot be denied

> that ideas differ among themselves, as the objects themselves do, and that one is more excellent than the other, and contains more reality, just as the object of the one is more excellent than the object of the other and contains more reality. And so to determine what is the difference between the human mind and the others, and how it surpasses them, it is necessary for us to know the nature of its object, that is, of the human body.[12]

Remarking that "it is not necessary for the things I wish to demonstrate" that a complete explanation of this scale of excellence be offered here, he confines himself to a general observation:

> [1] [I]n proportion as a body is more capable than others of doing many things at once, or being acted on in many ways at once [*aptior est ad plura simul agendum vel patiendum*], so its mind is more capable than others of perceiving many things at once [*simul*]. And [2] in proportion as the actions of one body [*unius corporis actiones*] depend more on itself alone [*magis ab ipso solo pendent*], and as other bodies concur

[11] Spinoza, *Ethics*, II Postulate I.

[12] Spinoza, *Ethics*, II 13 Schol.

> with it less in acting [*et quo minus alia corpora cum eodem in agendo concurrunt*], so its mind is more capable of understanding distinctly. And from these [truths] we can know [3] the excellence of one mind over the others, and also see the cause why we have only a completely confused knowledge of our body, and many other things which I shall deduce from them in the following.[13]

Let us look at those two "direct proportion" claims a little more closely:

1. **Simul Theory**: The more one body can act or be acted on in many ways *at once* or *together* (simul), the more its mind can perceive many things *at once* or *together*.

2. **Independence (of Actions) Up, Concurrence Down → Distinctness Up**: The more the actions of one body depend on itself alone, and the less other bodies "concur" with it in acting, the more its mind can understand distinctly.

It is evident that the proportions are meant to work in the other direction as well, i.e., [1] the *less* one body can act or be acted on in many ways at once, the *less* its mind can perceive many things at once, and [2] the *less* the actions of one body depend on itself alone, and the *more* other bodies "concur" with it in acting, the *less* its mind can understand distinctly. Besides the initial interpretive challenge of understanding these two claims together, the key thing to note in these two claims is the use of "simul" in [1] and the fact that [2] is specifically a statement about "one body" (*unius corporis*). As we will see, "simul" does a lot of work not just in Spinoza's definition of "one body" but also in Descartes's decisive definition of "one body" (within his definition of local motion). Both are thus claims that anticipate and call forth an account of what makes for the 'oneness' of one body. Another

[13] Spinoza, *Ethics*, II 13 Schol., my numbering.

key element in understanding the motivation of the "Physical Interlude" is that Spinoza is not just attempting to account for the fact that Nature is *scaled*—that some things "contain more reality (or perfection)" than others (see [3] above)—but also for *how* such distinctions emerge, make their mark, and leave the 'unity' or 'oneness' of Nature as a whole undisturbed. Immediately following these general reflections, Spinoza claims that in order to get clearer on the above matters, it is necessary to "premise a few things concerning the nature of bodies"—and with this the "Physical Interlude" begins.

The first portion of this interlude, we are told, is devoted to the "simplest bodies" (*corpora simplicissima*). The first two axioms inform us that all bodies either move or are at rest (A1′) and each body moves now more slowly, now more quickly (A2′). The first lemma tells us that it is through motion and rest, speed and slowness, that bodies are distinguished from one another—and *not as substances*. The demonstration asserts that the first part of this lemma is "known through itself," probably because it was a key tenant of Cartesian physics, and any reader of the time would have recognized, with the first couple of axioms and lemmas, that Spinoza is deep within Cartesian territory. In defense of the further claim that bodies are not distinguished *as substances*—a claim that is the first aspect of this account that would *not* have been "self-evident" to someone who accepted the basic tenants and definitions of Cartesian physics—Spinoza initially appeals to the famous IP5 (substance is unique) and IP8 (that every substance is necessarily infinite). But he adds that this aspect of the lemma "is even more clearly evident from those things which are said in IP15S." Proposition 15, of course, told us that whatever is, is in God, and nothing can be or be conceived without God. The Scholium to this proposition purports to "provide a fuller explanation" of the fact, proven in IP14Cor2, that God or Nature has extension for one of its infinite attributes, by taking his opponent's arguments to the contrary (i.e., that God is not extended), one by one (there are two in total). Because of their relevance in helping set the stage and better contextualizing the theory constructed in the interlude, I will briefly review the contents of that Scholium.

The first argument to the contrary provided by his opponents rests on the assumption that corporeal substance, as substance, consists of parts, from which they deny that it could be infinite, and so pertain to God, on the basis of certain "examples" that amount to *reductio* arguments that begin by assuming an infinite quantity to exist, and lead to the allegedly "absurd conclusions" that one infinity is greater than another. The second argument claims that God, as a supremely perfect being, cannot be acted on; yet corporeal substance, since it is divisible, can be acted on; therefore, his opponents conclude, extension cannot belong to the essence of God. Both arguments, Spinoza notes, rest on the same supposition: *that corporeal substance is composed of parts.* Spinoza remarks that IP12, that no attribute of a substance can be conceived from which it follows that substance can be divided, and IP13C, that no substance (and so, no corporeal substance), qua substance, is divisible, already suffice to demonstrate the error in this shared supposition. Further, the "absurdities" they reach in their arguments do not in fact follow from the supposition of an infinite quantity, Spinoza contends, but rather from the supposition that an infinite quantity is *measurable and composed of finite parts.* So, assuming their arguments were otherwise valid, the "absurdities" that follow merely show that an infinite quantity is not measurable and not composed of finite parts, claims Spinoza believes himself to have already shown to be true in IP12 and IP13. Corporeal substance must in fact be infinite (IP8), unique (IP5), and indivisible (IP12)—*not* composed of finite parts, many, and divisible.

Spinoza then compares the approach of these erroneous arguments to how

> also others, after they feign that a line is composed of points, know how to invent many arguments, by which they show that a line cannot be divided to infinity. And indeed it is no less absurd to assert that corporeal substance is composed of bodies, *or* parts, than that a body is composed of surfaces, the surfaces of lines, and the lines, finally, of points. All those who know that clear reason is infallible must confess this—particularly those who deny the existence of a vacuum. For if corporeal substance could be so divided that its parts were really distinct, why, then, could one part not be annihilated, the rest remaining connected with one another as before? And why must they all be so fitted together that there is no vacuum? Truly, of things which are really distinct from one another, one can be, and remain in its condition, without the other. Since, therefore, there is no vacuum in Nature, but all its parts must so concur that there is no vacuum, it follows also that they cannot be really distinguished, that is, that corporeal substance, insofar as it is a substance, cannot be divided.[14]

Again, at a pivotal moment, there is a reliance on a key tenant of Cartesian physics—that there is no vacuum or void (on which more below). Spinoza goes on to draw the distinction between quantity conceived "abstractly or superficially" by the imagination and quantity conceived "by the intellect alone," where the former leads us to believe that quantity is finite, divisible, and composed of parts, while the latter enables us to see that quantity must be infinite, unique, and indivisible. Apparently, this distinction is "plain to anyone who knows how to distinguish between the intellect and the imagination"; perhaps more helpfully, Spinoza notes that it is especially clear if one acknowledges that "matter is everywhere the same, and that parts are distinguished in it only insofar as we conceive matter to be affected

[14]Spinoza, *Ethics*, I 15 Schol.

in different ways, so that its parts are distinguished only modally, but not really."[15] The example he provides of this in the present context will turn out to be quite important:

> For example, we conceive that water is divided and its parts separated from one another—insofar as it is water, but not insofar as it is corporeal substance. For insofar as it is substance, it is neither separated nor divided. Again, water, insofar as it is water, is generated and corrupted, but insofar as it is substance, it is neither generated nor corrupted.[16]

In a sense, then, Spinoza's demonstration of the first lemma of the "Physical Interlude"—in referring us back to the Scholium of IP15—would seem to indicate that a body, *insofar as it is that body*, can be properly understood to have parts separated from one another and distinguished by their motion and rest, while it cannot be thus understood *insofar as it is substance*. In mentioning how water is water and not some other body, Spinoza is already looking forward to the account of what makes for the defining unity of a composite body. But so far, all we only know is that it will be by virtue of motion and rest, speed and slowness, that bodies are distinguished, not by virtue of some difference in substance.

The second lemma of the interlude, for all its brevity and seeming innocuousness, is perhaps the lemma that will get used the most frequently and with the most far-reaching consequences: *all bodies agree* [conveniunt] *in certain things*. In the second part of the interlude, devoted to composite bodies, the theory of "agreement" between bodies that emerges will become of central importance. However, because we are still dealing with the

[15]Ibid. As will be discussed in the next section, this idea is, again, fundamentally Cartesian (despite some of the rather un-Cartesian conclusions Spinoza will come to draw from it).

[16]Ibid.

"simplest bodies" at this point, the demonstration develops this "agreement" in necessarily very abstract or general terms:

> all bodies agree in that they involve the concept of one and the same attribute (by D1) [namely, extension], and in that they can move now more slowly, now more quickly, and absolutely, that now they move, now they are at rest.[17]

Since we are dealing with the simplest bodies—those distinguished only by motion and rest, speed and slowness—and thus since we are not yet able to attribute any other "higher-order" properties of organization that result from interactions between bodies (thereby developing more subtle or nuanced "agreements"), all that can be said about the agreement between these "simplest bodies" is that they all move or are at rest, and move with greater speed or slowness.

Lemma 3 articulates the idea that when a body moves or is at rest, it must have been determined to motion or rest by another body, which itself must have been determined to motion or rest by another, and so on, to infinity. A corollary of this is a "principle of inertia" (another key element of Cartesian physics), i.e., that a body in motion stays in motion until it is determined to rest by another body, and that a body at rest remains at rest until determined to motion by another body.

> When I suppose that body A, say, is at rest, and do not attend to any other body in motion, I can say nothing about body A except that it is at rest. [...] If, on the other hand, A is supposed to move, then as long as we attend only to A, we shall be able to affirm nothing concerning it except that it moves. If afterwards it happens that A is at rest, that of course also could not have come about from the motion it had. For from the motion nothing else could follow but that A would move.

[17] Spinoza, *Ethics*, II, Physical Interlude, L2 Dem.

> Therefore, it happens by a thing which was not in A, namely, by an external cause, by which [A] has been determined to rest.[18]

So a body that is in motion (at rest), considered in itself alone (i.e., "not attending to any other body in motion"), can only be said to be in some definite state of motion (at rest). The point here is that such bodies are distinguished only in terms of being in a definite state of motion and rest, and that is all we can say. If, later, it is said to come to rest (be in motion), this can only have come about from an external cause determining it to change its state.

Note that everything that has been said thus far in the interlude pertains to the simplest bodies. It is the simplest bodies that change only with respect to their motion and rest (and only from the action of an external cause), so that, considered on their own, they are simply in some state of motion or rest—and, as such, these bodies (regarded on their own) cannot be conceived of as changing. And yet, this is not to say that, in changing states (by being determined by an external body), such bodies cannot contribute in any way to how they are changed. The next axiom begins to offer some clarification on precisely this point:

> (A1″) All modes by which a body is affected by another body follow both from the nature of the body affected and at the same time from the nature of the affecting body, so that one and the same body may be moved differently according to differences in the nature of the bodies moving it. And conversely, different bodies may be moved differently by one and the same body.

So when a body in some definite state of motion or rest is affected by another body, this affection follows from the natures of the body affected and the affecting body. The sort of

[18]Spinoza, *Ethics*, II, Physical Interlude, L3 Cor.

thing Spinoza has in mind here, regarding the simplest bodies, reveals itself in the second axiom (which is more or less lifted out of his *Principles of Cartesian Philosophy*):[19]

> (A2″) When a body in motion strikes against another which is at rest and cannot give way, then it is reflected, so that it continues to move, and the angle of the line of the reflected motion with the surface of the body at rest which it struck against will be equal to the angle which the line of the incident motion makes with the same surface.

Why Spinoza chose to include just this particular law dealing with the collision of bodies, of the several such laws explored in the second part of *Principles*, is not entirely clear. However, I take it that it is meant to serve, in the present context, as a mere sample or prototype of the sorts of possible affective interactions between simple bodies suggested by A1″. Following this axiom, Spinoza remarks that all that he has said thus far "should be sufficient concerning the simplest bodies, which are distinguished from one another only by motion and rest, speed and slowness," drawing our attention back to the fact that the first four axioms and three lemmas are meant to apply first of all to the simplest bodies. Another possible reason for the apparent abruptness of the fourth axiom is that Spinoza seems eager to pass on from the "simplest bodies" to "composite bodies," which he now proposes to address—and indeed, his theory of composite bodies is where his greatest originality lies and where the greatest connections are made with many other claims of the *Ethics*.

In the next chapter, we will look closely at how, beginning in the second part of the "Physical Interlude," the *Ethics* develops a powerful theory of how composites come to form a unity that could be called an 'individual', or *one body*. But before doing this, we need to take seriously that the theory of "simplest bodies" on top of which this theory of composite

[19] See Spinoza, "Principles of Cartesian Philosophy," II 28.

bodies builds seems to be more or less lifted or inherited from Descartes. It is certainly true that for Spinoza, as was the case for Descartes, the simplest bodies are only distinguished by their motion and rest and can only change in one respect, namely with respect to their motion and rest. But even what 'motion' and 'rest' mean, why they are regarded as distinct states, how they are related, and how they contribute to *distinguishing* bodies—these equally seem to be best understood in the context of Descartes' physics. While Spinoza's theory of composite bodies forming unities, as introduced in the second half of the interlude, goes beyond Cartesian physics in a number of important ways, by beginning in the first part of the interlude with the nature of bodies considered in isolation—the *corpora simplicissima*, distinguished from one another as they are entirely in terms of definite states of motion and rest—Spinoza seems to expect that, at least as far as these issues are concerned, the reader is already on intimate terms with Cartesian physics. Fleshing out this underlying theory of the simplest bodies, then, can be accomplished by pairing a reading of the highly condensed first part of the "Physical Interlude" of the *Ethics* with Spinoza's presentation of the Cartesian account of bodies in states of motion and rest from the *Principles*. To the extent that the physics of the *Ethics* is innovative and offers repairs or improvements to the Cartesian scheme, these innovations can only be properly understood once one appreciates how much of the traditional Cartesian physics Spinoza takes for granted.

The Cartesian Account

As we are primarily interested in Spinoza's particular appropriation of elements of Cartesian physics, we can content ourselves with briefly paraphrasing just some of the most fundamental ideas of Cartesian physics:

1. **Space = Body**: space and body are one and the same, which has as an immediate consequence that *there is no vacuum* (no such thing as a space without a body).

2. **No Atoms**: atoms—indivisible particles—do not exist; extension is always divisible. This divisibility, moreover, is *real*.[20]

3. **Conservation of Motion in the Whole Universe**: the "quantity of motion" in the universe is conserved:

> Although...motion is nothing in moving matter but its mode, yet it has a certain and determinate quantity, which we can easily understand to be able to remain always the same in the whole universe of things [*semper in tota rerum universitate*], even though it be changed in its individual parts [*in singulis eius partibus mutetur*].[21]

4. **Neighborhood Definition of Local Motion**: motion is the transference of a part of matter or a body from the neighborhood of contiguous bodies that are in a state of rest, to the neighborhood of others.

5. **All Variety in Matter and Diversity of Forms Depends on Motion**:

> There is thus only one kind of matter in the whole universe, and this we know only by its being extended. *All the properties we distinctly perceive to belong to it are reducible to its capacity of being divided and moved according to its parts; and accordingly it is capable of all those affections which we perceive can arise from the motion of its*

[20] "And although we cannot comprehend by thought how this indefinite division takes place, we should not, however, doubt that it happens, because we clearly perceive that it necessarily follows from the nature of matter most evidently known to us [...]" (Descartes, "Principles of Philosophy - Part II," 35). Consider also: "So even if we were to imagine that God wanted to have brought it about that some particles of matter were reduced to a smallness so extreme that it did not admit of being further divided, even then they should not properly be called indivisible, for though God had rendered the particle so small that it was not in the power of any creature to divide it, he could not however deprive himself of the ability to do so, since it is absolutely impossible for him to lessen his own omnipotence, as was before observed. Wherefore, absolutely speaking, the smallest extended particle is always divisible, since it is such of its very nature" (Descartes, "Principles of Philosophy," II 20).

[21] Descartes, "Principles of Philosophy - Part II," 36, translation modified. But note that this does not, on its own, tell us how any individual bodies will behave and achieve their unity or persist as individuals.

> *parts.* For the partition of matter in thought makes no change in it; but all variety in matter, that is, all the diversity of its forms, depends on motion.[22]

As opposed to the medieval Aristotelian account of motion as a passage from one state to another, motion for Descartes is itself a state of a body, and one that persists; and all the diversity of forms in nature and all the affections ultimately derive from the different *motions among parts.*

6. **Rest and Motion are Distinct, Opposite States**: an irreducible distinction exists between a body at rest and a body in motion, i.e., a body at rest is in a different state entirely from one that just moves extremely slowly.

Each of these six components will enter Spinoza's thinking in a number of decisive ways. But, for the moment, the first thing to remark is that while Descartes denies the existence of indivisible particles, in analyzing simple motions he will also frequently speak of bodies as if they were in some sense "simple"; moreover, his cosmology ultimately seems committed to the existence of something like simple particles or "corpuscules." How he can do this becomes clear after closer inspection of perhaps the most pivotal of the above aspects of Cartesian physics, on which so much else depends: his neighborhood definition of local motion.

The Cartesian Account of Local Motion

Descartes begins his treatment of motion in his *Principles of Philosophy* by distancing himself from what he calls the "vulgar" or "common" sense of motion as "the action [*actio*] by which

[22] Descartes, "Principles of Philosophy," II 23, my emphasis.

some body passes [*migrat*] from one place to another."[23] There seem to be a few leading ideas motivating his singling out of the "vulgar" action-based conception of motion for attack, a conception he himself had been repeatedly tempted by in the past, especially prior to writing his *Principles*. Perhaps the most decisive of his reasons for opposing this "vulgar" definition would appear to be its formulation in terms of a change of *place*, for the determination of a given place is relative to a frame of reference; but since this frame of reference is arbitrarily chosen, the same thing can at once be said to both change its place and not change its place—a feature of the account that ultimately makes it an arbitrary matter whether or not a body is in motion: "For instance, someone sitting in a boat while it sails from port thinks himself to be moving if he looks at the shore and considers it to be unmoved; but not if [he looks] at the boat itself, among the parts of which he always maintains the same location."[24] Against this vulgar definition of motion, then, Descartes proposes his own definition of local motion, which allegedly accords more with "the truth of things":

> **Local motion** is the *transfer* [*translationem*] *of one part of matter or one body* [*unius partis materiae sive unius corporis*], *from the neighborhood* [*vicinia*] *of those bodies immediately contiguous to it* [*immediate contingunt*] *and considered at rest, into the neighborhood of others.* By 'one body' or 'one part of matter' I mean whatever is transferred simultaneously/together [*id omne quod simul transfertur*], even though this may in fact consist of many parts which have different motions relative to each other.[25]

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[23] Descartes, "Principles of Philosophy," II 24.

[24] Descartes, "Principles of Philosophy - Part II," 24.

[25] Descartes, "Principles of Philosophy," II 25, emphasis original, translation modified.

Despite what Garber calls the "almost baroque complexity" of the definition,[26] and in addition to any of its positive virtues (some of which will be seen in what follows), Descartes seems to believe that this definition is able to avoid the most devastating of the deficiencies he attributed to the "vulgar" action-based definition, and can distinguish motion as a transference from the force or action responsible for putting a body in motion. The main moving parts of his definition that must end up doing all the work of overcoming those deficiencies are the notions of (i) *transference* or *translation*; (ii) *one part of matter* or *one* body; and (iii) *contiguous neighborhoods*. One might think of each of these intertwined concepts as designed to address, respectively, the questions of (i) the *how*, (ii) the *what*, and (iii) the *where* of a motion.

In his exposition of the Cartesian account of motion in the *Principles of Descartes' Philosophy* (Part II, "Concerning the Physical World"), Spinoza more or less repeats eight of the nine definitions and all twenty-one of the axioms without commentary or modification; yet he offers extensive commentary on the (eighth) definition, that of local motion, providing five separate remarks on the definition. The bulk of these remarks are more or less derived from certain of Descartes' own elaborations on his original definition. Perhaps the most important of all these elaborations is Descartes' insistence on the *neighborhood* aspect of the definition, and its centrality in addressing the most decisive deficiencies in the "vulgar" definition:

> I have furthermore added that translation/transfer [*translationem*] takes place *from the neighborhood* [vicinia] *of contiguous bodies to the neighborhood of others*, but not 'from one place to another', because the

[26]In Cottingham, *The Cambridge Companion to Descartes*, 305.

> meaning of 'place' is varied and depends on our thinking. But, when we understand by motion that translation which occurs from the neighborhood of contiguous bodies, since only one [group of] bodies can be contiguous to the same mobile at the same moment of time, we cannot attribute to this mobile many motions at the same time, but only one.[27]

Many of the observations and qualifications contained in both Descartes' original elaborations on his definition and in Spinoza's presentation are vital to understanding the full import of the definition of local motion. For this reason, I paraphrase the most important observations surrounding this definition following Spinoza's own presentation in the form of five remarks.

1. **The What of Motion**: *A part of matter or* one body *is not necessarily without parts, but rather is whatever is transfered together (or at the same time, simultaneously).*[28]

   It may seem like a minor point, but it is this observation concerning *what* moves that most directly addresses the need to reconcile the non-existence of atoms with the repeated use of the notion of "parts of matter" or "one body" throughout Cartesian physics. In referring to a "part of matter" or "one body," Cartesian physics is not committed to the existence of a particle, if by that one means something without parts or an atom. The notion that is supposed to make this an acceptable line of thinking is the re-definition of a part of matter or one body as "whatever is transfered at the same time [or together, *simul*]." While the reader might be willing to concede the negative point—that a part of matter or *one* body need not be defined as something not having parts of its own—it would

[27] Descartes, "Principles of Philosophy - Part II," 28, translation modified.

[28] See Spinoza, *Principles of Cartesian Philosophy*, II Def8, Remark1; Spinoza, *The Collected Works of Spinoza, Volume I.*

be fair to wonder about just how informative or viable the positive definition of what makes a body 'one' is. We will be in a better position to address this after introducing the other four remarks. For now, note that the oneness of a body is not something that requires that it be "simple" or lack parts, and that Descartes would have us replace such an implicit alignment of 'oneness' and simplicity by an explicit identification of 'one body' and 'being transfered together'.

2. **The How of Motion**: *Transference*—"something that is always in moving bodies"—does not equal the force by which bodies are transfered. Moreover, regarding the force by which bodies are transfered: *force is required just as much to explain rest as it is to account for motion.*

Regarding the second point, Spinoza adds that it is "self-evident" that the force required to impart certain degrees of motion to a body that is at rest is also required to take away those certain degrees of motion from the body in order to bring it to rest. He remarks that this is "something that is proven by experience":

> For we use nearly the same force to put in motion a boat resting in still water as we use to stop suddenly the same boat when it is moving. The force would surely be exactly the same if we were not aided in stopping the motion by the weight and resistance of the water the boat displaces.[29]

On its own, this does not seem like a very important observation. However, it is observations of this sort that enable Descartes to regard rest as a state or mode distinct from that of motion (a position also adopted by Spinoza). After having made the same point that we are mistaken to think that more action or "conatus"

[29] Spinoza, *Principles of Cartesian Philosophy*, II Def8, Remark2.

(II, 26) is required for motion than it is in cases of rest (or arresting motions), Descartes added an important observation:

> Since it is not a matter of that action which is understood to be in the mover, or in that which stops motion, but of translation [*translatione*] alone and of the absence of translation, or rest, it is clear that this translation cannot be outside the moved body and that this body is in one mode [*modo*] while it is transferred [*cum transfertur*] and in another when it is not transferred, or when it is at rest; with the result that the motion and rest in it are nothing other than two different modes [*duo diversi modi*].[30]

Here we see that, whatever else "translation" or "transference" is meant to be or accomplish, it is "not outside the moved body," and if a body is being transferred (is in motion), it is in one mode (or way), while if it is not transferred (is at rest), it is in another mode (or way). So the presence or absence of *transference*, not some moving or arresting *action* coming from outside the body, is what creates a *modal distinction* in bodies. And recall that, for Descartes, it is distinctions of motion and rest that alone account for all the diversity of forms and affections in nature.[31] In order to better understand this notion of transference and how it achieves what the "vulgar" account does not, the next two remarks, devoted to the 'where' of motion-as-transfer, focus on further specifying the important notion of *neighborhood*.

3. *Transference is not made from one* place *to another, but from the neighborhood of contiguous bodies to the neighborhood of other contiguous bodies*:

[30]Descartes, "Principles of Philosophy," II 27, translation modified.

[31]See (5) in the section above, "The Cartesian Account."

> For place is not something in the object, but it depends upon our thought, so much so that the same body may be said at the same time to change its place and not to change it; but not at the same time to be transferred from the neighborhood of contiguous bodies and not to be transferred. For only one body at the same moment of time can be contiguous to the same moving body.[32]

4. *Transference is not from* any *neighborhood of contiguous bodies, but specifically from the neighborhood of those contiguous bodies that are at rest.* Elaborating on the observation that transference is not "absolutely" from a contiguous neighborhood but specifically from the neighborhood of those contiguous bodies at rest, Spinoza remarks that the motion of one body with respect to another is, despite how we may ordinarily think about things, *reciprocal*. Descartes had elaborated on this distinction as follows:

> For this translation is reciprocal, and body AB cannot be understood to be transferred from the neighborhood of body CD unless it is understood at the same time that body CD is also transferred from the neighborhood of body AB; and clearly the same force and action is required from the one part as from the other. Whence, if we should want to assign to motion a nature altogether its own and not relative to something else, we should say that, when two contiguous bodies are transferred, one in one direction, the other in another direction, and thus are mutually separated, there is as much motion in the one as in the other. But this is very much incompatible with the common way of speaking; for, since we are accustomed to stand on the earth and consider it as at rest, even though we see some of its parts contiguous to other smaller bodies transferred from the vicinity of those bodies, we do not, however, therefore think the earth to be moved.[33]

[32]Spinoza, *Principles of Cartesian Philosophy*, II Def 8, Remark3.

[33]Descartes, "Principles of Philosophy - Part II," 29.

In a marginal note Descartes made in his copy of the *Principles*, Descartes offers some clarification of this point, saying that "Nothing is absolute in motion except the mutual separation of two moving bodies," while "that one of the bodies is said to move, and the other to be at rest is relative, and depends on our conception."

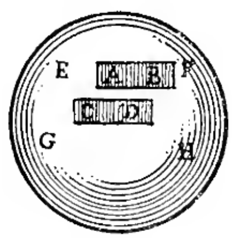
> Motion and rest differ truly and modally [*modaliter*] if by motion is understood the mutual separation of bodies and by rest the lack [*negatio*] of this separation. However, when one of two bodies which are separating mutually is said to move, and the other to be at rest, in this sense motion and rest differ only in reason [*ratione*].[34]

In endorsing the "reciprocity of transfer," Descartes makes it clear that "we must remember that everything that is real and positive in bodies that are moved, according to which they are said to be moved, is also found in others contiguous to them, which nevertheless are only viewed as at rest."[35] The point of all this is that a given body can only partake of one motion, for the body is either at rest with respect to, or "translating" away from, its nearest neighbors. Yet, of two contiguous bodies separating from one another, whether we *say* that one (and not the other) is moved is indeed a purely "relative" matter. Does this "relationism" not make it a purely arbitrary matter whether a body is even considered as at rest or in motion, thereby making Descartes' account succumb to the same criticism Descartes waged against the "common" or "vulgar" conception of motion? Descartes attempted to foreclose such an objection with reasoning of the following sort:

[34]Cited in Cottingham, *The Cambridge Companion to Descartes*, from Adams-Tannery, Vol XI, 656-57.

[35]Descartes, "Principles of Philosophy - Part II," 30.

> The chief reason for this is that motion is understood to be of the whole body [*totius corporis*] that is moved, and thus it cannot be of the whole earth in the case of the translation of some of its parts from the neighborhood of smaller bodies to which they are contiguous, since one may often show many translations of this sort, mutually contrary, on the earth. For example, if body EFGH is the earth and on it at the same time body AB is transferred from E toward F and CD from H toward G, although the parts of the earth contiguous to this body AB are transferred from B toward A, and the action in them should not be less nor of another nature for this translation than in the body AB, we do not therefore, understand the earth to be moved from B toward A, or from west to east, because, by the same argument, in the case where its parts contiguous to body CD are transferred from C to D, it would be understood that it was also moved in the other direction, that is, from east to west, which two [motions] contradict one another. Thus, lest we draw back too much from the common way of speaking, we should not say here that the earth is moved, but only bodies AB and CD; and thus for other bodies. But in the meantime we will remember that everything that is real and positive in bodies that are moved, according to which they are said to be moved, is also found in others contiguous to them, which nevertheless are only viewed as at rest.[36]



The point—a rather consequential one to which we will return in detail after discussing the final remark—is that motion as the mutual separation of a body from its contiguous neighbors is not something that can be reduced to a matter of our thinking: either a body and its nearest neighborhood are separating from one another or they are not. In this way, with respect to the same contiguous neighborhood, a body is always either in transference or not—never both—and

[36]Descartes, "Principles of Philosophy - Part II," 30.

so motion and rest must be distinct modes of body. The aspect that does depend on our thinking, though, is that when a body is in motion, strictly in terms of the neighborhood definition, one must say that a body is separating from its neighborhood and that its neighborhood is separating from it—in this mutual separation, "there is as much motion in the one as there is in the other,"[37] something that leaves open the possibility that we say the body x moves with respect to its resting neighborhood Y or the neighborhood (now regarded as a body) moves with respect to x (now regarded as its resting neighborhood). This should all become clearer in the section that follows the next remark. For now, suffice it is say that *motion is a neighborhood phenomenon.*

5. *Each body has only one motion* proper *to it*:

> from the definition it is clear that each body has for itself only one motion proper to it, since only in regard to contiguous resting bodies is it said to depart. Nevertheless, if a moving body is a part of other bodies having other motions, we clearly see that it is also able to participate in countless other motions. Yet because we cannot easily understood so many motions at once, or even recognize all of them, it will suffice to consider in each body that one motion proper to it.[38]

In the original, Descartes is a little clearer, or at least more explicit, on this point than Spinoza:

> But even if any body has only one motion proper to it, since it is understood to recede from only one [group of] bodies contiguous to it and at rest, it can nevertheless also participate in innumerable others, if, for example, it is a part of other bodies having other

[37]Descartes, "Principles of Philosophy - Part II," 29.

[38]Spinoza, *Principles of Cartesian Philosophy*, II Def 8, Remark 5.

> motions. For example, if someone walking in a boat carries a watch in his pocket, the wheels of his watch will be moved with only a single motion proper to them, but they will participate also in another insofar as, joined to the walking man, they compose with him one part of matter [*unam cum illo materiae partem component*]; and in another insofar as they are joined to the boat in a heaving sea; and in another insofar as they are joined to this sea; and finally in another insofar as they are joined to the earth itself, if indeed the whole earth be moved. And all these motions are really [*revera*] in these wheels; but, because so many [motions] cannot be understood at the same time, nor also can all be known, it will suffice to consider in itself that single [motion] that is proper to each body.[39]

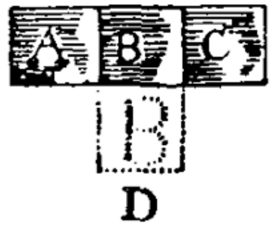
To this pivotal observation, Descartes immediately added that, conversely, a single motion "proper" to some body can sometimes be considered as many—for instance, "when we distinguish in the wheels of chariots two different motions, to wit, one circle about their axis and another along the length of the path through which they are borne."[40] Yet while this process of separating one motion into many parts may be "useful" in enabling us to "perceive it more easily," nevertheless, "absolutely speaking, one should count only one motion in any body."[41]

Where we are ultimately headed with all this is to an explanation of how it is that motion and rest can be what makes a body 'one' or a unity, while the definition of motion appears to already make appeal to "one body," without this account being "viciously circular." To better see what is going on in the above definition with its many qualifications, we will follow Descartes' and Spinoza's habit of using diagrams such as those seen in the previous passage or in Spinoza's discussion of Proposition 8, in which he refers to the diagram

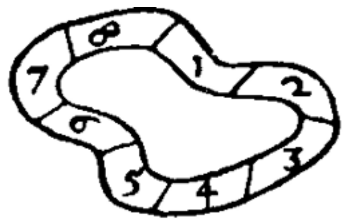
[39] Descartes, "Principles of Philosophy - Part II," 31.

[40] Ibid., 32.

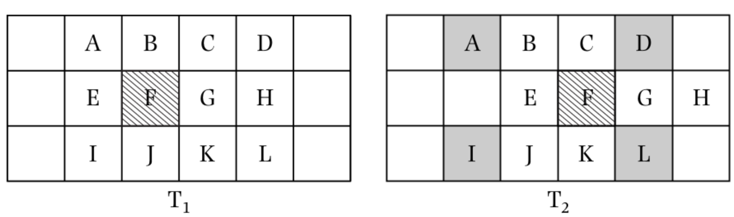
[41] Ibid.



or throughout his repeated discussions of "circles of moving bodies," referring to the diagram



In order to get a better sense of what is meant by the three moving parts of the definition of local motion—transference, *one* body, and neighborhoods of contiguous bodies—and to start unpacking the five observations on this definition, I will focus the discussion through this sort of diagram or representation. It is useful to begin by noting already how another commentator gives the following (instructively mistaken, as we will see) interpretation of what Descartes had in mind with his definition, referring to the same sort of diagram:



> Consider the body F. Given Descartes' definition of "motion," it is in motion from T_1 to T_2 because when we regard its neighborhood—bodies A, B, C, G, K, J, I, and E—as being at rest, we find that F changes its neighborhood over that time, for at T_2 F is no longer surrounded by A or I, but by new bodies, D and L, in addition to bodies from the previous neighborhood, B, C, G, K, and E. Of course, given the way the definition reads, we can regard F as being at rest and arrive at a

> very different result: each of the bodies that surround F at T_1 are in motion from T_1 to T_2 because they change their neighborhoods.[42]

The interpretation of the otherwise faithful representation given by Sowaal misses or forgets to consistently apply a number of important aspects of the definition. Given that Sowaal goes on to use her condensed interpretation of this representation—specifically, the interpretation involved in the final sentence of the above quote—to ultimately reinforce the standard accusation of "vicious circularity" in Descartes' definition, and even, in the end, (supposedly so as to "save" it from such alleged circularity) to argue that the "reciprocity of transfer" makes motion ultimately dependent on the role of a perceiver (thereby making Descartes an "idealist"), it will be important to sharply differentiate my interpretation from Sowaal's and spell out the true implications of the Cartesian account of local motion (especially as it relates to Spinoza).[43]

Because it will turn out to be natural to consider the contiguous neighborhoods of our chosen cell's nearest neighbors, we will work with a diagram that includes a somewhat larger grid of cells than that displayed in Sowaal's diagram, one that includes the neighborhoods of the neighbors of our chosen cell. In multiplying the cells, for ease of reference I have numbered them, instead of using letters (after which re-labeling, the cell of interest, namely

[42]Sowaal, "Idealism and Cartesian Motion," in Nelson, *A Companion to Rationalism*, 251-52.

[43]Sowaal's intepretation is simply chosen as representative of a more general tendency in the literature, in the sense that most commentators on Descartes seem to agree about the charge of circularity, but also seem to have thought about these matters no more than what is indicated in Sowaal's paragraph above, which is perhaps in part responsible for the more or less general consensus that the definition of local motion is ultimately "viciously circular."

F, becomes 15).[44] Consider, then, following Sowaal's example, one possible motion during one time-step, shifting 15 to the right :

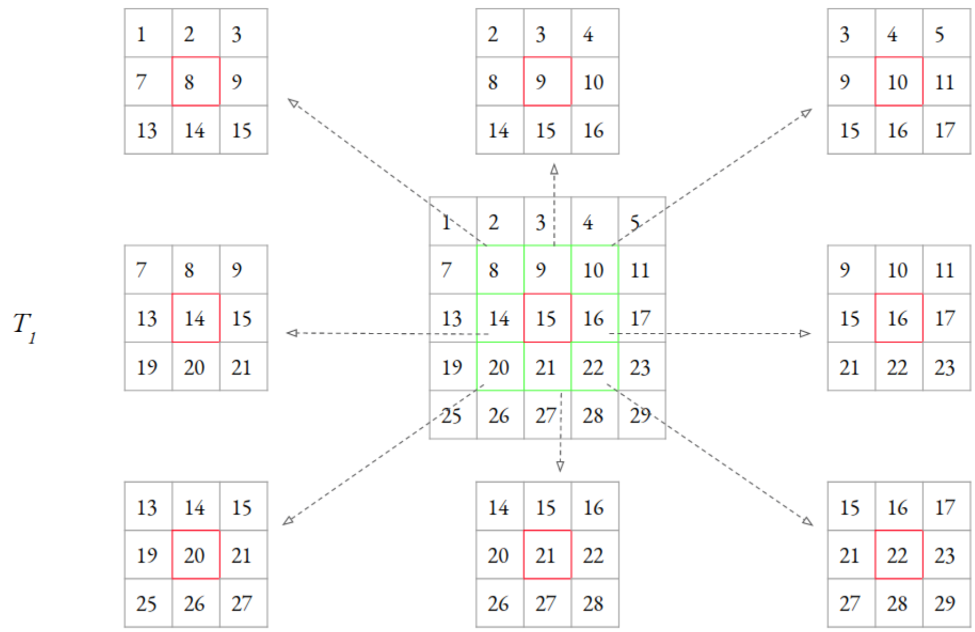
1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

T_1 $\xrightarrow[\text{right}]{\text{shift}}$ T_2

1	2	3	4	5	6
7	8	9	10	11	12
...	13	14	15	16	17
19	20	21	22	23	24
25	26	27	28	29	30

In terms of Descartes' definition of local motion, what can we say is really going on here? Let us make some elementary, but important, observations. First, recall that we know from the definition that *one body* = 'that which, in neighborhood transfers, gets transferred together/simultaneously'. We know, moreover, that *motion* = 'the transference of one body from its neighborhood of immediately contiguous bodies (considered as at rest) into another neighborhood' = 'transfer of *whatever gets transferred together* from its neighborhood of immediately contiguous bodies considered as at rest into the neighborhood of other bodies'. We can also emphasize a couple of more general points. At any given time, a body x has a neighborhood that consists of those bodies that are touching it—call this its neighborhood of nearest neighbors, or *nearest neighborhood* for short. Suppose you are "playing" the cell $x =$ 15. At any given moment (on any given "turn"), you have a definite neighborhood of nearest neighbors, namely those 8 neighbors or cells that immediately surround you (highlighted in green in the above diagram). But each other cell, including each of your own neighbors, has its own, distinct nearest neighborhood (in this case, of 8 cells):

[44]Note that these numbers do not "mean" anything and should not be taken to refer to any sort of real quantity or numerical assignment to the cell representing a body; the numbers are just labels.



Now, observe that if y is in the neighborhood of nearest neighbors of x, then x is in the neighborhood of nearest neighbors of y (and vice versa). (This is the *symmetry* of the nearest neighborhood relation.) If y is in the neighborhood of nearest neighbors of x, and z is in the neighborhood of nearest neighbors of y, it is not necessarily the case that z is in the neighborhood of nearest neighbors of x. (This is the general *intransitivity* of the nearest neighborhood relation.) Moreover, from one time to another, a body x can change some of the neighbors in its neighborhood of nearest neighbors, while others do not change. And when a body x's neighbors change, they can change with respect to x in two ways: in their membership in the neighborhood of nearest neighbors (ceasing to be a nearest neighbor) or in their position in the neighborhood of nearest neighbors (remaining a nearest neighbor but altering their relative position in the neighborhood structure). Those neighbors Z in the neighborhood of nearest neighbors of some x that do *not* change from one time to another in relation to x have changed neither their membership in the neighborhood of nearest neighbors, nor their position (relative to x) in the neighborhood of nearest neighbors.

Thus, if x moves in relation to some resting contiguous neighbors, given that the neighbors included in Z did *not* change in relation to x, those neighbors comprising Z necessarily moved *together with* x in relation to the remaining, resting neighbors. Therefore, for such times, Z *is transferred together with or at the same time as* x—and so in fact forms part of the *one body* comprised of x and Z, on account of the definition of 'one body'. But given that a body always has a neighborhood of nearest neighbors at any given time, the one body comprised of x and Z has (say) Y for its neighborhood at one time and some other ordered collection of bodies, (say) Q, for its neighborhood of nearest neighbors at another time. And thus, from one time, T_1, to another, T_2, the one body comprised of x and Z changes its neighborhood of nearest neighbors *together* (or *simultaneously*) from Y to another neighborhood Q, i.e., they move as one body in relation to those of their nearest neighbors that, other than the changes required by the symmetry of neighborhood change, rest. In sum, then, bodies that change their neighborhoods together are in motion in relation to those that, together, do not change their neighborhoods.

Now let us make some particular observations about the example motion with which we began:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

T_1

$\xrightarrow[\text{right}]{\text{shift}}$

1	2	3	4	5	6
7	8	9	10	11	12
...	13	14	15	16	17
19	20	21	22	23	24
25	26	27	28	29	30

T_2

First notice that, at T_1, 15's nearest ("contiguous") neighbors (including 15 itself) are given by $\{8, 9, 10, 14, 15, 16, 20, 21, 22\}$. At T_2, 15's nearest neighbors are $\{9, 10, 11, 14, 15, 16, 21, 22, 23\}$.

These neighborhoods are highlighted in green. While some of the cells that were contiguous neighbors of 15 at T_1 are no longer nearest neighbors of 15 at T_2—specifically, 8 and 20—and some of the nearest neighbors of 15 at T_2 were not nearest neighbors of 15 at T_1 (namely 11 and 23), this mere change in the collection of nearest neighbors from one time to another is not what is of primary concern. Moreover, note that it is not the case that 11 and 23—the "new" neighbors—are simply "substituted" for 8 and 20 either. And even certain of the neighbors that *remain* nearest neighbors of 15 throughout T_1 and T_2 have changed their position relative to 15 in the transition. These neighbors remain neighbors of 15 throughout the transition from T_1 to T_2, but strictly in terms of the overall aim of understanding local motion in terms of the changes to the nearest neighborhood systems—which are ordered structures—it is vital to realize that (in this example) *they are not the same neighbors from one time to the next*, for their different arrangements in relation to 15 at T_1 and T_2 makes them part of distinct neighborhood systems.

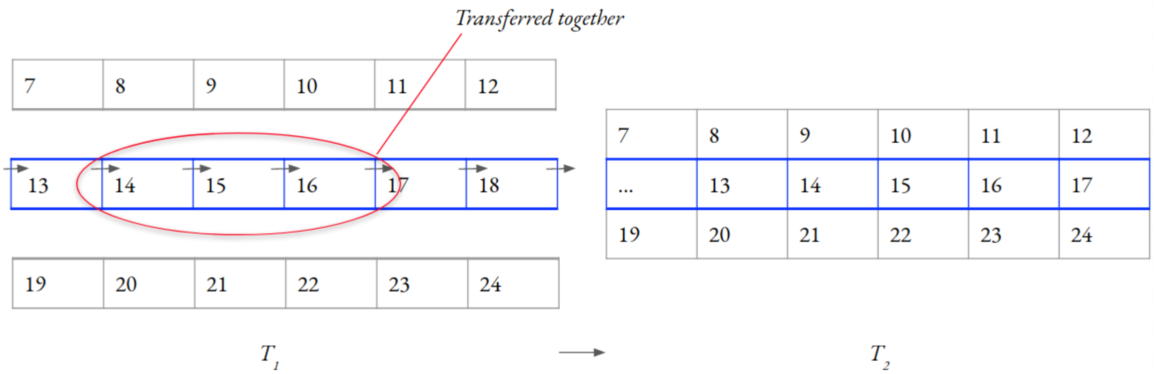
On the other hand, there *are* certain neighbors of 15 that, by contrast with the nearest neighbors that persist through both times but not in the same position in the neighborhood systems, not only persist as neighbors through the transition, but even retain the same relative position within the neighborhood structure of 15—namely 14 and 16. As far as the system of nearest neighbors are concerned, this unit 14-15-16 is the only thing that remains invariant throughout T_1 and T_2.

8	9	10
14	15	16
20	21	22

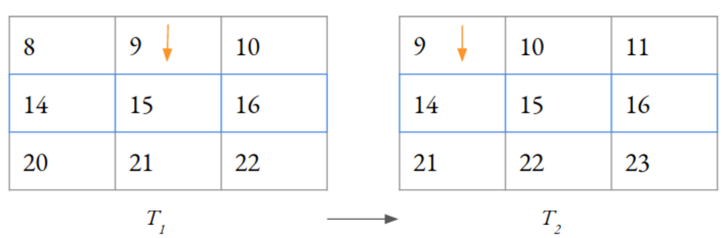
T_1 → T_2

9	10	11
14	15	16
21	22	23

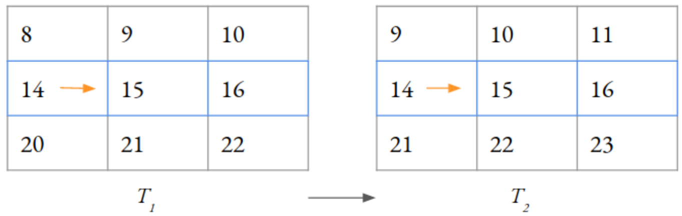
But really, the important thing to realize is that these particular nearest neighbors do not "remain unchanged" in some abstract or absolute sense; more precisely, they remain unchanged in relation to 15 precisely by virtue of being transferred together with 15.



As mentioned above, other cells—specifically 9, 10, 21, and 22—appear in the neighborhood structures of 15 at both T_1 and T_2. However, these cells may convey different motions to 15 at T_2 than they would at T_1. For instance, for simplicity, consider a simple motion brought about through "a body pushing another," and "pushing" in a straight direction. At T_1, a downward motion of 9 on its own (i.e., all else being equal) may effect a downward motion of 15, but at T_2 any downward motion of 9 on its own could only effect a downward motion in 14, not 15:



At T_2, if 9 wants to convey a downward motion to 15, it can only do so indirectly and in conjunction with certain motions from certain of the other neighbors. By contrast, at T_1 a horizontal motion of 14 or 16 may lead to a corresponding left-right motion communicated to 15 (and any strictly vertical motion of 14 or 16 would leave 15 unaffected, all else being equal), and the same thing holds with respect to horizontal motions communicable to 15 at T_2 (as well as the indifference of 15 to strictly vertical motions transmitted by 14 or 16).



This begins to explain why it is a matter of the *ordered* neighborhood structures and not the simple abstract collection of neighbors; in a different relative position, the same neighbor may convey different motions to the same body (one of its neighbors) and the same motions to different bodies (one of its neighbors). We need only look at those cells or neighbors for which a given motion that would have been *conveyed* with a certain result to certain neighbors at T_1 would now, at T_2, convey that same given motion with a different result or a different motion to that same neighbor—this is equivalent to looking at those cells that have different neighborhood configurations at T_1 and T_2. And, anticipating Spinoza's treatment of composite bodies in terms of invariance in the communication of the motions among the "number of bodies" that form its parts, this equivalence will turn out to be rather significant.

Such basic observations already suffice to expose a first serious problem with Sowaal's own interpretation: note that she remarked that "when we regard [F's] neighborhood—bodies A, B, C, G, K, J, I, and E [this was $8, 9, 10, 16, 22, 21, 20$ and 14 in our representation]—as being at rest, we find that F [15 in our version] changes its neighborhood over that time. . . ." The first issue here is that it simply does not make sense, in regarding the neighborhood of 15, to regard *all of* $8, 9, 10, 16, 22, 21, 20$ and 14 as "being at rest," as she proposes we do. Rather, it only makes sense to regard *either* all of $8, 9, 10, 20, 21, 22$ as at rest *or* all of $14, 15, 16$ as at rest. This is not an idle point. For, from our initial observations, which simply follow the neighborhood notion and what makes for 'one body', it is immediately clear that it also does not in fact make sense to regard 15 *as a single unit* or as 'one body'. It is evident that, in the hypothetical motion proposed by Sowaal, the sub-neighborhood consisting of the horizontal band that includes 14-15-16 (and possibly extends further to the left and right) is to be regarded as an irreducible unit, as *one body.*

Instead of observing this, Sowaal chooses to focus on how F is "surrounded by new bodies. . . in addition to bodies from the previous neighborhood." But, following the definition alone, this change in the abstract set of neighbors (which cells belong to a set of neighbors) is simply not the information that concerns us. Again, this is not an idle point. Moreover, this description leads Sowaal to claim that, "of course, given the way the definition reads, we can regard F as being at rest and arrive at a very different result: each of the bodies that surround F at T_1 are in motion from T_1 to T_2 because they change their neighborhoods." The problem is that it is not that "each of the bodies" surrounding F are in motion *because* they change their neighborhoods, especially if by neighborhoods we simply mean an unstructured or unordered list of neighbors (as Sowaal clearly, if implicitly, intends). The neighborhood notion is, importantly, about neighborhood structures, relations of relative position, not about lists of neighbors. More importantly, in attempting to "regard F as being at rest," in contrast to what Sowaal claims we learn, we first learn that *we cannot regard simply F as being at rest*, which forces us to consider F to be a part of the larger unit $\cdots$-14-15-16-$\cdots$, which we may then regard as at rest (or in motion). What in fact emerges here is *what is not an arbitrary choice of a perceiver.* The definition of local motion constrains and exactly specifies, in each transition from one moment to another, what is to delimit 'one body'. It is true that, with respect to this larger unit of which F is a part, we can regard the larger unit as at rest (or in motion), and the bodies surrounding this band as in motion (or at rest). This is because of the fact that while they could convey certain motions at T_1 to certain neighbors, in principle they must now convey *different motions to those same neighbors* (for their own neighborhood structures are different at T_2) and *the same motions to different neighbors*. One could equivalently regard the unit $\cdots$-14-15-16-$\cdots$ as in motion relative to the contiguous neighbors surrounding it, now regarded as at rest. But let us look more carefully at what exactly this "reversal of perspectives" actually involves.

If you are "playing" 15 at T_1 and you want to move rightward (assuming everything else stays as it is, for simplicity), you can only do so by having your nearest neighbors to the right and left give up their current neighborhood configurations. But this immediately entails, because of the symmetry of the "neighborhood" relation, that any of the neighbors of these nearest left-right neighbors will have to change their neighborhood system in part (specifically in the part that relates to your nearest neighbors). Notice that at T_2, to appreciate all that is involved in the "move" that has been made, we do not have to look at the grid of cells as if it were some "absolute space," tracking which "cells" (now understood to have some fixed place) have been assigned new labels. But superimposing a concept of "absolute space" is the only way it could possibly make sense to perceive any difference between

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

T_1 $\xrightarrow[\textit{right}]{\textit{shift}}$ T_2

1	2	3	4	5	6
7	8	9	10	11	12
...	13	14	15	16	17
19	20	21	22	23	24
25	26	27	28	29	30

and

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30

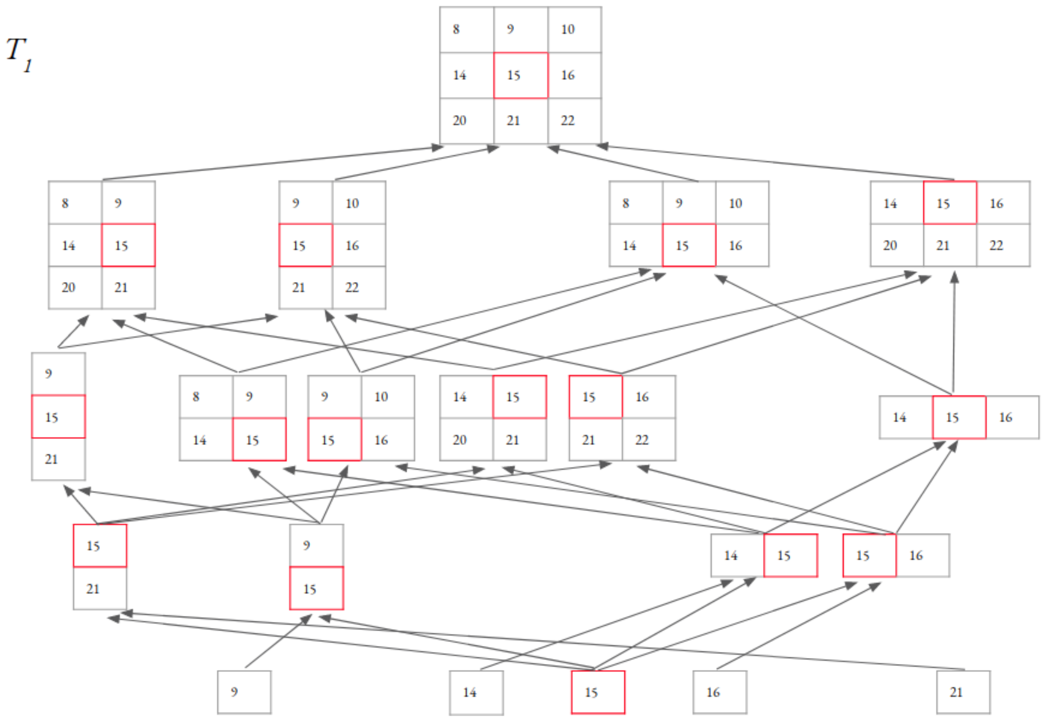
2	3	4	5	6	...
8	9	10	11	12	...
13	14	15	16	17	18
20	21	22	23	24	...
26	27	28	29	30	...

$T_1 \longrightarrow T_2$

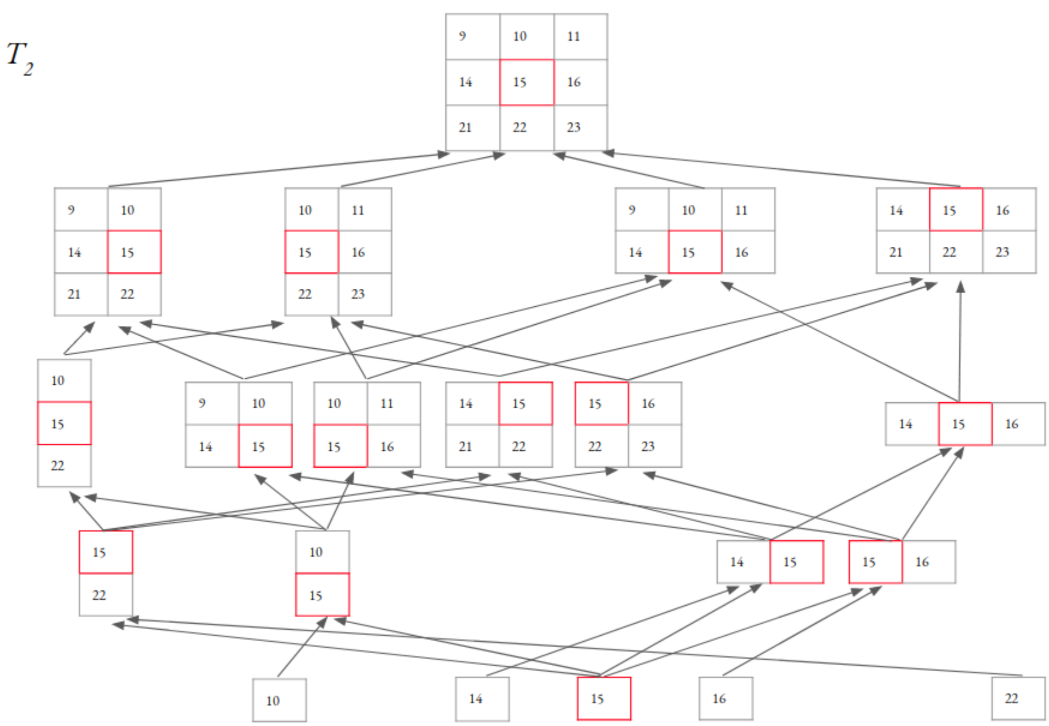
Notice that strictly in terms of Descartes definition, there is absolutely no way of detecting a difference between any aspect of the two presentations. Not just 15, but each of all the neighbors of 15, have exactly the same (nearest and extended) neighborhood structures at both times across each of the presentations. Yet the first presentation is what it would look like if we chose to regard 15 (and all that moves together with it) as in motion in relation to its nearest neighbors, regarded as at rest; while the second presentation is what it would look like to regard 15 (and all that rests together with it) as at rest, in relation to which its top and bottom nearest neighbors are regarded as in motion. The "reciprocity of transfer" that allows us to regard a body as in motion relative to a neighborhood of bodies regarded as at rest or regard it as at rest relative to the same neighborhood of bodies (now themselves regarded as in motion) is, importantly, not something that could ever have any effect on which bodies were to be regarded as *one body* by virtue of being transferred (or resting) together. Moreover, such a change in presentation or perspective could not do anything to change the fact that there is a mutual transference of neighborhoods from one time to the next, which results in a mutual separating of bodies. Motion was defined as *one body* (something the identity of which is not affected in any way by the perspectival change) being *transferred from its contiguous neighborhood into another neighborhood* (a change that is not in any way altered or undone by the perspectival change)—so nothing about the "reciprocity of transfer" will entail the sort of problematic "relativity" of motion and rest, if by that we mean that it jeopardizes our ability to say that, truly, a body is in motion and not not in motion. Such a charge would only have worked if we had been able to attribute to Descartes a notion of "absolute space" which is, pointedly, not his. Likewise, notice that by adding "chains" of motion-rest pairs, i.e., regarding a moving body's resting neighbors as themselves in motion with respect to other resting neighbors which are themselves in motion with respect to, etc., we still do not do anything to undermine our ability to really attribute motion to a body (unless you sneak back in a concept of absolute space).

Show me those neighbors that move and rest together with you (or with whom you move and rest) and I will show you *who you are*—this is one thing this definition tells us. But perhaps the most important thing to notice is this: in discovering this, we are really (or equivalently) discovering exactly those of your neighbors that, as your ability to convey or be determined by certain motions in a certain way changes, themselves change (in the same way) in their ability to convey or be determined by those same motions. It is precisely this aspect of the account of invariance or changelessness—something that, it should be emphasized, applies to bodies that *move*—that makes the determination of the identity or unity of bodies meaningful as more and more complicated interactions between many bodies are built up. Note however that, on its own terms, the determination is notably not *extrinsic*, not ultimately one that needs appeal to "external bodies" in specifying what counts as 'one body'; rather, it refers to those parts or neighboring bodies that remain in the same relative position with respect to one another and so continue to convey and be determined by the same motions in the same way.

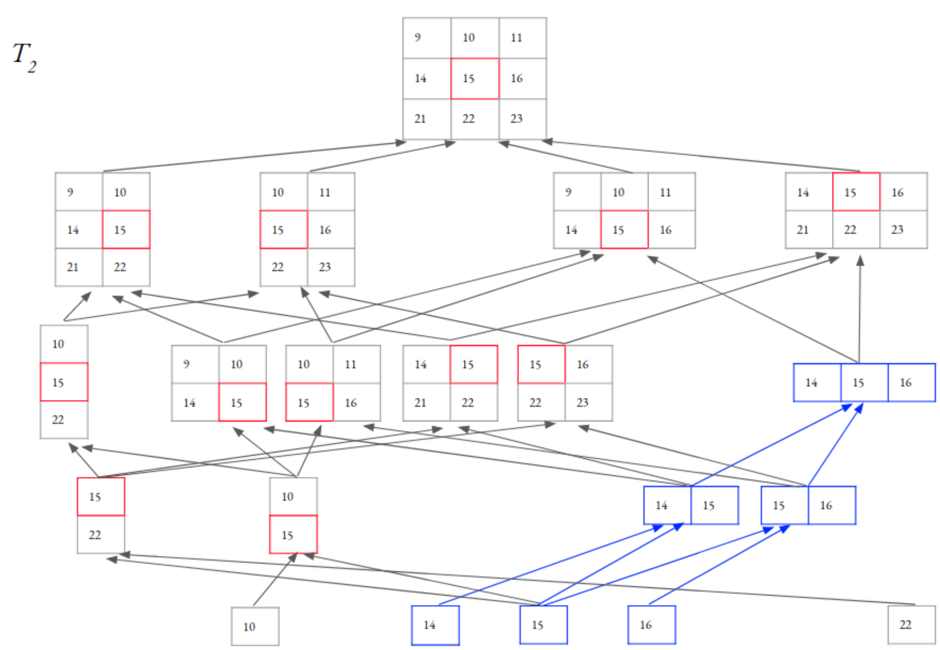
One lingering question is to what extent this neighborhood-transference definition applies more generally, beyond the simple case of colliding bodies changing direction. I leave the details for another time, and roughly indicate how this could be developed in terms of the sub-neighborhoods of the neighborhood system formed by the contiguous neighbors of 15, say, ordered among themselves by arrows indicating inclusion. If you then begin by considering certain bodies, such as 15, in isolation, it will emerge that without yet knowing anything about the motions given to each of 15's neighbors, certain sub-neighborhoods of 15's neighborhood system might, in principle, take on certain motions without this assignment directly affecting what motions can be assigned to 15. In other words, for certain sub-neighborhoods of the nearest neighborhood system, at least one assignment of motions to each cell group should exist such that the motions of the cell 15 can be made independently from, or unconstrained by, each those assignments. At each "level"—e.g., considering just the 1-sub-neighborhoods or taking each of the contiguous neighbors individually—there will be a way of assigning motions that does not directly constrain which motion gets assigned to 15. One can then remove from consideration those sub-neighborhoods of the nearest neighborhood of 15 that do not directly constrain the possible motions assigned to 15, leaving something like:



We could then look at the reduced sub-neighborhood structure of 15 at another time, T_2. For the sake of concreteness, we can look at our running example from before. At T_2, the corresponding sub-neighborhood diagram looks like:



Notice that we have said nothing about direction or put any restrictions on the nature of the motion. There is nothing about this way of thinking about things that requires that we restrict our attention to simple directional motions. The same reasoning should hold if we simply regard the motions being assigned to each cell-group at each level as some sort of generic motion, assignments that can constrain other assignments made to certain cell groups and be made independently of others. Note that in the above two diagrams, we have merely reproduced the sub-neighborhood networks corresponding to each neighborhood systems of 15 at two different times, and dissected these neighborhood systems along purely structural lines. Yet, in comparing the two diagrams of 15's nearest neighborhood's sub-neighborhood structure at T_1 and T_2, it can be seen that the only part of the diagram that has not changed is, again, the portion corresponding to the sub-system of neighborhoods of 14-15-16 (highlighted in blue):



So, without having said anything about what sort of motion we are considering, we could recover from this purely structural presentation—involving only information about the neighborhood systems of 15 at both times—the fact that the motion assigned to 15 cannot be regarded in isolation or independently from the motions assigned to the whole unit 14-15-16. While the same analysis could be performed on the other cells, looking at their nearest neighborhood systems, thereby discovering which of the other neighbors form a single unit and which move independently of one another, we have again found the appropriate level at which to regard 15: it is a part of the single unit 14-15-16. And this means two things. First, it means that the real neighborhood of contiguous neighbors of 15 is to be found at the level of the neighborhood of 14-15-16, and not of 15 alone. Second, it means that if you form a part of *one body*, this is equivalent to there being a certain mutual dependence or certain constraining between the motions available or assignable to you and those available to the other parts of that one body. Again, this could equally be interpreted in terms of how the parts continue to convey, and be determined by, the same motions in the same way.

While the reader might be somewhat distracted by the above presentation in terms of the sub-neighborhood structures, the decisive thing to realize here is that *all* the information encoding a specific motion of 15 is already contained in the two reduced *neighborhood structure* diagrams for T_1 and T_2. This means that all we need in order to specify how 15 has moved is specify the neighborhood systems at T_1 and at T_2. Given such information, we can further discover that with which it moves (and rests) as *one* and that which moves (and rests) independently of that unit. Moreover, this gives us a way of finding out *which level* is the right one to look at in considering the motions of an arbitrary part of matter, which bodies move and rest as 'one'. On the other hand, once we perform this analysis for each of the many bodies surrounding 15, we are presented with the choice, in comparing these neighborhood systems, of taking the unit 14-15-16 to be resting with respect to its contiguous neighbors or taking its contiguous neighbors to be resting with respect to it. But, again, such a "perspectival" decision does nothing to undermine the informative invariants that emerge in this neighborhood approach.

The reader who has not yet fully appreciated the power of this neighborhood-transference definition of motion, and its role in beginning to account for the individuation of individual bodies, should realize what this definition does: (1) it is purely about *structure* (neighborhood systems), specifically local structures (*nearest* neighborhood systems), certain parts of which change while others remain the same; as such, (2) it renders unnecessary any notion of "absolute space" or reliance on any notion of a fixed place; yet at the same time (3) it provides a way of non-arbitrarily determining the limits of *one* body and unequivocally stating whether or not it is 'one' body; and (4) it takes seriously, and accounts for, the "reciprocity of transfer," for the fact that we can switch our frame of reference. Altogether, this is already a major accomplishment.

On the whole, while the "one body" we have been discussing was said to be not without parts (it may even be infinitely divisible), the interactions we have been considering have been simple changes of motion and rest. But already in this account there began to emerge a sense of invariance articulated in terms of how the parts of a body continue to convey motions between themselves in a fixed manner, without this stability ever threatening to dissolve distinctions between these parts. Notice how in speaking of "cells" and treating them as individuals, we have not presupposed the existence of indivisible atoms, yet at the same time it is not some "optical illusion" we were performing in treating them as individuals—for, recall, to be a "particle of matter" or "one body" is not to be without parts, but to be a stable way in which one's parts change their nearest neighbors together.

On the Charge of Circularity

Various commentators have referred to the Cartesian definition of local motion as "circular," even "viciously circular," and others have accordingly used this to imply that Spinoza's own definition of an *individual* composite body in terms of its "motions and rests" is itself subject to the same, or similar, charge.[45] For instance, regarding Descartes' definition of local motion, Sowaal writes:

> [It] involves a serious logical flaw: it is circular. "Motion" is defined in terms of a body's changes with respect to neighboring bodies, and "one body" is defined in terms of that which is in motion.[46]

I think it is fair to say that if such charges against Descartes' definition of local motion were correct, the same charge could be made against Spinoza. However, I believe that it can be shown that these (intimately related) charges are based on a deep misunderstanding or lack of appreciation of the full force and subtle construction of the original definition of local motion.[47] More importantly, this issue is really involved in a whole network of conceptual problems, many of which seem to recapitulate, at various levels, the serious interpretive problem facing the understanding of the relation between Descartes' definition of local motion and his definition of 'one body'.

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[45]See, e.g., Slowik, "Descartes' Physics"; Nolan, *The Cambridge Descartes Lexicon.*

[46]Sowaal, "Idealism and Cartesian Motion," in Nelson, *A Companion to Rationalism*, 254.

[47]Moreover, it is worth noting that while commentator after commentator refers to the problematic circularity of this definition, it is very rare that anyone bothers to spell out how that "circularity" is supposed to work: they simply assume that Descartes defines motion in terms of bodies and bodies in terms of motion—as this is indeed what a superficial glance at the words of the definition may suggest—and that this is, well, *bad.* I believe that, properly understood, there is no problem here—in fact, what these commentators take to be a "circularity" is a virtue of the definition.

Purely on the surface, it is apparent how one might be inclined to think that the definition of 'one body' (packed within the definition of 'local motion') purports to be explained by what it presupposes, thus giving credence to the accusation of circularity. But this perception misses the meaning of the terms as they are defined, and relies on a tendency to impose on Descartes certain assumptions that are not his. For instance, Sowaal's pat summarization of the definition above is mostly a straw-man, and is undercut by closer examination of how the concepts of neighborhood, transference-together, and transference of neighborhoods are linked together. If the terms of the definition are applied strictly, one observes not "circularity," but a consistent, if unnervingly economical, definition. One observes that while the first part of Sowaal's characteristic claim is basically correct, the second part—that "'one body' is defined in terms of that which is in motion"—is either too vague to be a useful summary of a demanding definition, or simply misrepresents the actual definition.[48] In replacing the necessary work of unfolding the complex and subtle conceptual interlinkages between the definition's key notions of *one body, neighborhood, transference/translation* and *transference together (simul)*, with an assessment of its legitimacy at a purely *verbal* level, commentators have not only too quickly dismissed what is serious and original in the definition, but they have robbed themselves, in advance, of the means of explaining Spinoza's theory of the unity of *composite bodies.* In addition to the positive virtues of the definition that enable it to

[48]In the context of presenting this definition, phrases like "in terms of" that which is in motion are too underdetermined to be useful. It would be more accurate to say that 'one body' is defined as comprising whatever components that, in transferring their nearest neighborhoods, transfer together (and so, in relation to one another, *do not move*, i.e., do not change their neighborhood, but in relation to remainder of their nearest neighborhood, *do move*). In saying that "one body" is defined "in terms of what is in motion," Sowaal's formulation is either simply wrong or too vague to be evaluated.

avoid the charge of vicious circularity, the main error in the standard charge of circularity consists in the mistaken imposition of two anachronistic assumptions:

- attempting to regard a motion from a fixed point of view external to the motion-rest complex, i.e., presupposing (against Descartes' own express commitments) a notion of "absolute space" or at least a notion of space as distinct from body. This error is related to a reversion to the erroneous assumption that space is something that could "contain" a body; against which assumption Descartes is explicit that space is body and that bodies are local regimes of transition. Space "as a whole" is thus nothing more than the infinite interlocking of local regimes of transition. Descartes' bodies do not go from one fixed *point* or *place* to another; and they are not *presupposed*. His bodies are complexes of neighboring parts that achieve local stability in transiting and resting together—a body is formed as the result of certain parts of matter having connected up in a certain way with certain of its neighboring parts, so that when either changes its neighborhood (i.e., the relative positions of the parts of the configuration), or resists changes to its neighborhood, the other parts do the same. A body is thus nothing more than a specific conformity of parts to a stable dynamics of neighborhood transitions.
- attempting to force the characteristic invariance of individual bodies to be either (i) a static property of simple bodies (bodies without parts), if it is intrinsic, or (ii) something purely *extrinsic*. Rather, invariance of a body emerges as a dynamic regime of continuity established among the local transitions of parts—only in this way is a body *whatever components, in moving locally, move together, and*

> *in resting, rest together* (and so, viewed "internally," remain *invariant* in their manner of conveying motions and rests among one another).

More directly, the entire force of the charge of circularity rests on a poorly-thought-out understanding of what a *neighborhood* is and how the neighborhood-transference actually works. To see this, let us return to the paired definitions of local motion and 'one body':

Definition 4.0.1. Local motion is the *transfer* of *one body* (or part of matter) from the *neighborhood* of those bodies immediately contiguous to it and considered at rest, into the neighborhood of others. By '**one body**' is meant that which is transferred together/simultaneously, even though this may in fact consist of many parts which have different motions relative to each other.

I take it that the charge of circularity can be summed up thus: the definition appears to define *motion* as one body's change of neighborhood (i.e., change of which bodies are contiguous to it), and then defines 'one body' as that which changes neighborhood; so if x is one body, then x is 'that which changes neighborhood', but a motion of x is itself defined as a change in x's neighborhood; therefore, a motion is 'a change in neighborhood of that which changes neighborhood'.[49] There are a number of very serious problems with such a presentation. Before ending this chapter, let us take them one by one, presenting things in the form of a back-and-forth dialogue:

[49] This is how many commentators usually present it also, for instance: "The problem, of course, is that Descartes has defined motion as a change of contiguous bodies, and then proceeds to define body as that which moves (translates, transports). Although this circularity threatens the entire edifice of Cartesian physics [...]" (Slowik, "Descartes' Physics").

DESCARTES: Do you agree that by 'motion' I mean that what I call 'one body' (we can return later to whether I have a right to refer to such a thing) changes its nearest neighborhood (regarded as at rest) for another nearest neighborhood?

CRITIC: Yes, that is what you say. But...

DESCARTES: Well, hold on a second. Before you get too excited, do you understand exactly what is involved in this change of nearest neighborhood?

CRITIC: Well, yes, I think I do. A body is always flanked by neighbors—which are just other *bodies*, since there is no void—some of which are touching it. The collection of those neighbors that are touching it forms its *nearest neighborhood*. In changing its nearest neighborhood at one time for another nearest neighborhood at some other time, a body is said to be in motion. But what I have a problem with is ...

DESCARTES: Hold on a second. On the surface, I see no problem in anything you have said thus far. The issue, however, is in the problematic *vagueness* or *equivocation* inherent in your penultimate sentence: 'In changing its nearest neighborhood for another'. It seems to me that you have not bothered to unpack how this 'change of nearest neighborhood' works. So let us consider this more closely. Do you agree that a body always will have a nearest neighborhood?

CRITIC: Yes, of course—no void, and all that.

DESCARTES: Good. Now, something has to be said about what is meant by 'neighborhood'—and, in fact, we can build off of your own remark about the non-existence of a void. First, I hope you would agree that whatever assumptions we choose to make about dimensionality and the shape or figure of bodies, as long as we are granting that the concept of neighborhood makes sense, we are thinking that any given body is *near* certain bodies and *farther* from others. Is that fair?

CRITIC: That seems hard to dispute. Where are you going with this?

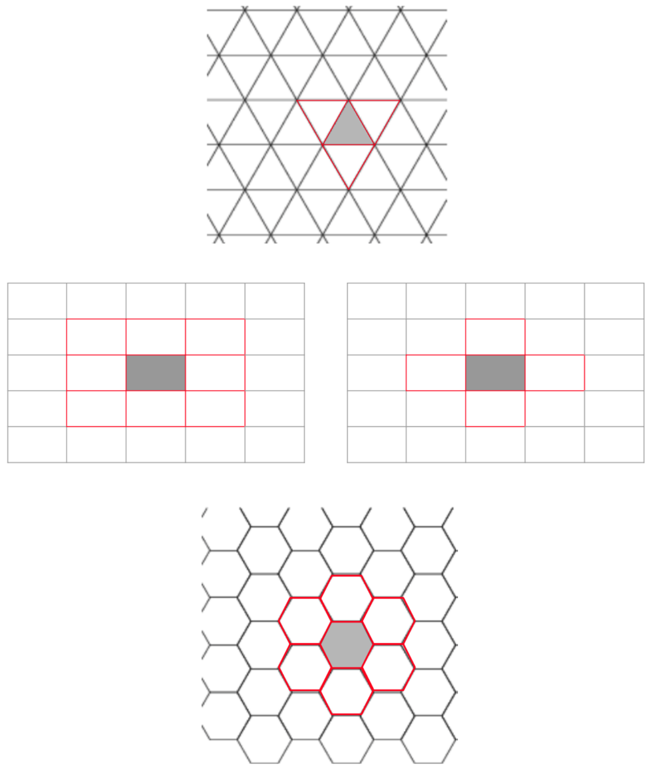
DESCARTES: Bear with me—we are just getting started. Before rushing to conclusions, we should be sure we agree about what this notion of 'neighborhood' involves and what it does not. Do you agree that—without having to make any commitments about the dimensionality of space or fundamental 'shapes'—if we were to try to conceptualize this notion of neighborhood in one-dimension, it would "degenerate"?

CRITIC: Sure, that makes sense. In one-dimension, there are no 'shapes' at all, so one body of one shape could not be surrounded by other bodies of the same shape or other shapes. And so we could not get the account off the ground to speak of certain bodies as being near others.

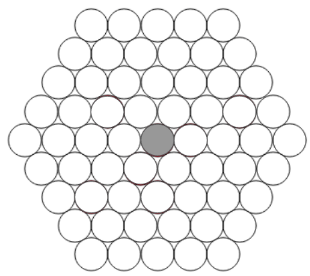
DESCARTES: Good. I suppose then, in thinking of neighborhoods, we have in mind representations that are at least two-dimensional? (Again, I am not asking you to commit to saying that extension is of a certain dimension or that bodies are of a certain shape. All I am asking is if, in forming an intelligible concept of 'neighborhood', there is already inherently a certain minimal requirement that we have an understanding of this concept as "doing work" in at least two-dimensions.)

CRITIC: Fine, I think that seems okay.

DESCARTES: Presumably, then, in thinking of neighborhoods, at least in the back of our minds we are working with representations of the following sort: [*Gets out a pencil and sketches the following figures.*]



You get the idea. Different neighborhoods can be chosen, for each shape, but tesselation with triangles is as crude as we can get; and, in that case, the smallest number of neighbors we can get away with using to form a nearest neighborhood is of course three. As we consider shapes that have more sides, we will accordingly need more neighbors to form a nearest neighborhood. We could, of course, use cells with different shapes, so that there is no restriction that the shape of one's neighbors is the same as the chosen cell. But, even in that case, we are not going to be able to get away with less than three neighbors. And note that as we pass to higher dimensions, this minimal requirement on the number of neighbors in our neighborhood will only increase (for any given tesselation pattern). The point in all this is to realize, though, that what we *do not* have in mind is a picture like this:



The problem, here, is that with such a hypothetical tesselation by circles—polygons with infinite sides—there will be 'voids', something you recognized we cannot admit. If, on the other hand, we treat those little left-over 'wedges' as bodies of their own, we will still have that a nearest neighborhood will have at least 3 neighbors.

CRITIC: Okay, I get it. Where is all this going?

DESCARTES: The first thing it tells us is that in speaking of a neighborhood of any given body, we are necessarily thinking of *more than one* (in fact, more than two) neighboring bodies.

CRITIC: And that is informative why?

DESCARTES: Earlier you agreed that a body always has a nearest neighborhood. Now, if a body changes its nearest neighborhood, then since you saw that its nearest neighborhood at any given time will include at least three neighbors, do you agree that in general such a change must involve certain nearest neighbors ceasing to be nearest neighbors or ceasing to be in the same relative position in the nearest neighborhood structure of our body?

CRITIC: Certainly.

DESCARTES: Well, do you agree that nothing demands that in general, when *certain* nearest neighbors change in one of the above ways, *all* of the nearest neighbors change? What I mean is this: in general, while certain neighbors change in any change of neighborhood, cannot certain other neighboring bodies in the neighborhood remain nearest neighbors of our body or even remain in the same relative position with respect to our body through this change?

CRITIC: Well, yes, I suppose nothing prevents certain of the (at least three) nearest neighbors of our given body from changing while others do not. And, I think I see where this is going: in such a case, we would of course still have a *neighborhood change* or *transfer of neighborhood*. It fact, it seems this sort of thing may even be typical.

DESCARTES: That is true. So, to summarize, in changing neighborhoods, a body can change certain neighbors while others can remain the same neighbors (i.e., in the same relative position in relation to our body). In the extreme particular case where all of a body's nearest neighbors change, I suppose you would agree that we could still say that one thing had not changed its relative position with respect to our body—

namely the body itself. So... what we have seen is that when a body changes its nearest neighborhood structure, certain sub-structures can remain the same (by which we mean have the same role in our body's neighborhood structure) while certain others can change.

CRITIC: Fine. But what does all this have to do with *motion*?

DESCARTES: Perfect. That is where we are going with this, and now is exactly the time to address it. How did we define a motion again?

CRITIC: One body changes or 'transfers' its nearest neighborhood for another nearest neighborhood.

DESCARTES: Aren't you forgetting something?

CRITIC: Let me see... Oh yes! The nearest neighborhood from which the one body transfers to another was taken to be *at rest*.

DESCARTES: Good. And that is important, as we'll see. Suppose, then, we have a body that transfers from one nearest neighborhood, taken as at rest, to another nearest neighborhood. We saw a moment ago that in a neighborhood transfer in general, *through* such a change, certain portions of the nearest neighborhood structure can be preserved while others can change. Take, then, those portions of the neighborhood structure of our given body that do in fact change through the transfer. Since we have been taking our body to be in motion, the definition requires that these just mentioned portions of the neighborhood structure be regarded as at rest, correct?

CRITIC: That is correct.

DESCARTES: Then, I ask you, what about those portions of the neighborhood structure, those neighborhood sub-structures, of our given body that are preserved through the transfer?

CRITIC: You will have to be clearer about what you are asking.

DESCARTES: Fair enough. I meant to ask: as for those certain portions of the neighborhood structure that remain invariant (in relation to our given body) through the transfer—do you say that these are at rest or in motion? (For I suppose you agree that all bodies are either at rest or in motion.)

CRITIC: Well, they are surely not at rest in the way those portions of the neighborhood structure of our given body that *do* change through the transfer are held to be at rest. In fact, in relation to the latter bodies, those portions of the neighborhood structure that remain invariant in relation to our given body must be said to *move*.

DESCARTES: That is exactly right. And that makes sense, right? After all, we are supposing that our given body is in motion—thus, if there are portions of its neighborhood structure that remain invariant in relation to that given body, we would expect that they would have to be in motion as well.

CRITIC: It's like they move *together* or *at the same time*.

DESCARTES: That is an interesting way to put it. Does it remind you of anything?

CRITIC: Well, yes, I suppose in such a case, we would look at whatever 'moved together' and say that this was 'one body'.

DESCARTES: Indeed. And you said that this 'one body'—as that which gets transferred together—was in motion? Specifically, that it was in motion in relation to those other portions of the neighborhood structure that remained at rest?

CRITIC: Yes, and it seems to me like we could equally have held this 'one body' to be at rest while the other portions of the neighborhood structure were regarded as in motion. In that case, the 'one body' would have been that which *rests together*.

DESCARTES: I am okay with that. The important point, however, is that in either case, *there is a motion*, specifically of our one body separating from those other portions of the neighborhood structure, i.e., there is a reciprocal change (and the *same* change, either way you look at it) in certain portions of the neighborhood structures.

CRITIC: That sounds reasonable.

DESCARTES: Now, I want to go back a few steps, for this point about the reciprocity of the transfer is not actually vital to your worries. A moment ago, I asked you if 'that which gets transferred together' (our 'one body') was in motion, and you agreed. Presumably you agreed because you had accepted that 'what gets transferred together' had a very definite sense, namely as picking out those sub-structures within the nearest neighborhood structures that remain the same from one time to another, that complex of nearest neighbors that preserves their same relative position, and so continues to convey the same motions among its parts from one time to the next; you had also presumably accepted that we can say exactly which portions of the neighborhood structures are *not the same* from one time to another, and that the *difference* in the two nearest neighborhood structures from one time to the next will in fact constitute the motion itself, the separation of those neighborhood sub-structures that remain the same from those that do not.

CRITIC: That sounds about right.

DESCARTES: I take it that, earlier, when you were worried about 'circularity' in the definition, you were worried about just this: that we appear to have to say that a motion is just a change to the nearest neighborhood structure of a given body, while we have also been saying that *a body* is just that complex of nearest neighbors that preserves the same relative position among themselves—which entails, among other things, that the components of this complex are able to convey the same motions in the same way to one another from one time to the next, which is just to say that they are able to convey the same nearest neighborhood changes in the same way. So it appears that we are saying that a motion is the first change in (some portion of) the neighborhood structure of whatever does not change its internal neighborhood structure.

CRITIC: Yes, more or less. It still seems like there is some circularity here.

DESCARTES: I can understand how you might think so. But consider this: are the nearest neighbors of my nearest neighbors necessarily *my* nearest neighbors?

CRITIC: No, I guess not. I guess there's a kind of in-built intransitivity to this whole 'neighborhood' business.

DESCARTES: Yes, there is. Now, another question: if there is a change in my nearest neighbor's nearest neighborhood structure, does this require a change in my relation to my nearest neighbor, i.e., in my nearest neighborhood structure?

CRITIC: Well, no, the 'nearest neighbor' relation is in general intransitive, which would seem to prevent any necessity from falling to what you described.

DESCARTES: Good. Then in accounting for a single motion I will not need to 'run through' an infinite chain of such motions—the chain is, as it were, broken after two steps. Observe that the implicit "logic of neighborhoods" is what allows for invariance to emerge in nature, while simultaneously requiring that this invariance emerge *amidst motion*, that it come about in the very way motions get propagated. If you are still not seeing how all of this prevents there from being any "vicious circularity" here, consider an analogous case. Suppose I decide to define a thing as all of its relations (or some sub-set of its relations). In then considering relations between things, we would be considering relations between relations. Do you believe that this is "viciously circular"?

CRITIC: Well, I suppose I am not entirely confident either way. It sure seems like it could be problematic.

DESCARTES: Suppose you told me about a trip you took, how long it took you, how much it cost, etc. If I responded to you in the following way, would you understand me? "It took you 10 hours and 500 dollars to get from New York to Chicago?! I could have gotten from Paris to South Africa much more quickly and cheaper!"

CRITIC: Yes, that of course makes perfect sense. We do this sort of thing all the time.

DESCARTES: We do. And do you perceive what exactly we are doing when we do this sort of thing? Just focusing on *how long the trip took*, notice that we are considering the *difference* between the data (length of trip, cost, etc.) assigned to one system of relations (specifically, involving itineraries or paths starting from New York and landing in Chicago, perhaps passing through other cities) and that assigned to another system of relations (involving itineraries starting from Paris and ending in South Africa). We have a *relation* between certain assignments of values—where these "assignments" are themselves a relation!—to each of two distinct systems of relations. In short, with such a mundane example, we already have a relation between certain relations between two systems of relations. Would you say that this process is "viciously circular"?

CRITIC: I suppose not. But I will need to process this a little more to see how this solves *my* problem!

DESCARTES: Good, do that.

The previous dialogue began to indicate how the charge of circularity derives from a confused or inconsistent application of the concept of *neighborhood.* To the extent that either of the accounts (Descartes' or Spinoza's) are circular, they are *recursive.* And this feature of the definition, in fact, is what makes for its power. Most important of all, if we were to (erroneously) find some circularity in the definition and accordingly attempt to "overcome" it, we would rob ourselves of the resources needed to defend Spinoza's chain of reasoning, through which he builds on Descartes' definition, that by composing compositions of compositions of . . . , we ultimately arrive at the fact that nature as a whole is one, an individual composite body, characterized by an invariant pattern in the communication, among the parts, of their motions. By contrast, preserving and appreciating the recursive nature of the definition of the body-motion complex will allow us to provide a consistent and compelling cause for this sort of reasoning, following the typically rationalist fashion. Only a definition such as this—of a recursive nature—is equipped to manage the subtle ontology of parts-wholes required by the rest of Spinoza's thoughts on the composition of bodies.

One final remark: in this context, by "recursive" we mean that in the definition the interpretation of a part of that definition requires the repeated application of the whole, something that parallels an underlying rule or action or procedure a part of which requires the repeated application of the whole. Such definitions are perfectly valid, and are not "circular" (if that means logically suspect) in general. It would be in danger of being circular in a logically suspect way if there were no "base case," i.e., an item that satisfied the definition without being defined in terms of the definition. In speaking of "simplest bodies" (which are not, however, without parts), I take it that Spinoza is providing just such a base case; in positing the existence of certain "corpuscles" (again, not indivisible), I take Descartes to be providing such a base case.[50] These are not bodies that are without parts, but rather bodies that are nothing more than some definite state of motion or rest. Nothing more can be said of them, and no more complicated interactions are taken into account. When Spinoza speaks (tellingly) of the "simpl*est* bodies," he is not implying that there are "simple" or "non-composite" bodies, if by that we mean atoms. We will have to look at the details of a given motion-rest complex in order to say which bodies were to be taken as 'one' and which moved and rested independently of one another. Whenever we consider the effects of the motions of 'one body' *as a whole* on some other 'one body' *as a whole*, we can speak of "simplest bodies."

On account of the importance of nearest neighborhoods in the definition of motion and the bodies that move, as we pass from bodies only distinguished by some definite state of motion and rest to consider "higher-order" interactions between bodies, these will continue to be constrained—though less and less so, until certain thresholds of compositeness are reached—by the locality of effects (i.e., by the decisive alignment of causality and closeness). As bodies become more and more composite, the "closeness requirement" can be relaxed more and more. This will allow for consideration of more and more "mediated" transmissions of effects, specifically for *patterns* by which motions are communicated (among bodies that are

[50] It is not necessary to present an account of Descartes' understanding of corpuscles at this time.

now, in principle, only mediately "close"). As it turns out, not only is it true that every body that our body will ever come into contact with in our existence will be composite, but these bodies will frequently be *highly* composite. In passing to consideration of composite bodies (and more and more composite bodies), as we do in the next chapter, we will see the peculiar nature of the recursive understanding of 'one body' and motion re-emerge at each "higher level." In this way, in building off of the Cartesian understanding of motion and 'one body', Spinoza is able to preserve a universe in which "closeness" and causality are allied, while allowing for ever more mediated and complex forms of unity. Thus, in terms of previous chapters, we see Spinoza propose a powerful *merger* of the Aristotelian tradition of continuity-as-closeness with the Oresmian notion of morphological continuity.

Chapter 5

Spinoza's Theory of Composite Bodies, and Common Notions

The Physical Interlude Again

After characterizing some of the fundamental properties of the simplest bodies in the first portion of the interlude, Spinoza proposes to "ascend" to the consideration of composite bodies, and offers the main definition:

> Definition: When a number of bodies, whether of the same or of different size, are so constrained by other bodies that they lie upon one another [*invicem incumbant*], or if they so move, whether with the same degree or different degrees of speed, that they communicate their motions to each other in a certain fixed proportion [*motus suos invicem certa quadam ratione communicent*], we shall say that those bodies are united [*unita*] with one another and that they all together [*omnia simul*] compose *one body* or *Individual*, which is distinguished from the others by this union of bodies.[1]

We know, of course, that Spinoza's "monism" requires that different things are not distinguished from one another as different *substances*, so each body must be made the 'one body' it is in some other way. With the above definition, we are given just those conditions (for any body or extended thing), for it is precisely by this particular manner of achieving "union" that each body will be "distinguished from others." To begin to unpack the definition, consider that we are dealing here with "a number of bodies" coming to form a unity or 'one body', and so we can be sure that such unities are not "simple," if that means without parts, for the entire definition is phrased in terms of a relation between "a number of bodies" or parts. The first "prong" of the definition is that a number of bodies, of possibly differing

[1] Spinoza, *Ethics*, II, "Physical Interlude," Def; Spinoza, *The Collected Works of Spinoza, Volume I*, my emphasis.

size, are constrained by other bodies so as to lie upon one another (or be "pressed down" on one another). With this particular requirement—namely that a number of bodies be so constrained by other bodies to lie upon one another—the definition initially restricts our attention to bodies that are *near*, i.e., putting us squarely within the purview of the neighborhood logic underlying the Cartesian account. As a short-hand, I referred in the previous chapter to this first condition in the definition as the "closeness requirement": regarding those bodies in particular that are nothing more than some definite state of motion or rest (and may in particular be at rest), those that are constrained by other bodies to be near, to be neighbors, are capable of forming *one body.* The second component of the definition is, by far, the more innovative, and applies above all to bodies in motion (with possibly differing degrees of speed). We can call this the **Fixed-Pattern-of-Communication Theory**: A body is said to be *one body* (or Individual) on account of the way various bodies ("a number of bodies"), each in some state of motion and with some degree of speed, *communicate* their motions to each other in a certain fixed manner (or pattern or ratio). More specifically, that which is invariant or fixed (*certa*) in the way with which the motions of various bodies are communicated between one another is what makes for the oneness or unity of any body. Part of the power of this definition, as we shall see, lies in how, with its emphasis on the patterns of communication of motions between parts, it captures a *dynamic* and *relational* sense of identity, while also ensuring the unequivocal and stable nature of what is being defined by requiring that this unity be a matter of *invariance.* While this definition applies to all bodies, to make things a little more concrete, note that it is something that appears particularly well-suited to the description of phenomena like sound waves. A sound is a pattern of disturbance of particles that propagates through a medium, where the disturbance always propagates by particle interactions between particles and certain of their nearest neighbors. (Here we have the "closeness requirement," always operative at the level of "local" interactions.) A definite sound is just a repeating or periodic pattern of the fluctuations in pressure caused by local particle interactions traveling through a medium. There is a regularity and invariance to how these definite sounds propagate such pressure fluctuations, yet this pattern is not static but is a matter of a transmission of the particularly regular way the *motions* of a number of interacting bodies are communicated among one another. These regular patterns combine with one another (and with other bodies), amplifying and canceling or dampening one another. In combining different sound waves, the sounds form literal *ratios* that behave exactly as mathematical ratios do. In being a fixed pattern in the communication of motions among a number of bodies, 'one body' is a certain consistent manner in which a number of bodies correspond or communicate with one another. Note that as this will in principle allow a number of bodies (not necessarily contiguous) to now have their motions communicated to one another through certain intermediaries, thus "lifting" or "relaxing" the closeness requirement, the closeness requirement never ceases to operate at the level of simple or local interactions between the component parts. A sound might be a regular pattern of propagation of the motions of particles—which, as such, may thus now involve, altogether, particles that are *not close*—but such motions are still always propagated step by step, from one particle to another, i.e., locally. In other words, even as the fixed-pattern-of-communication component of the definition covers more complex and mediated interactions, it does not *abandon* the closeness requirement (at the level of local and simple interactions). In short: the "or" [*vel*] of the definition is not to be read as *exclusive*.

After stipulating that the particular ways in which the parts or component bodies of an individual "lie upon one another" over a large or small surface—"so that they can be forced to change their position with more or less difficulty"—determine the difficulty with which changes in shape (figure) can be brought about, thereby giving rise to hardness, softness, and fluidity in bodies, the remainder of the interlude (lemmas 4-7) is devoted to determining what sorts of changes do *not* bring about change in an individual or destroy the fact that it is the *one* body that it is, i.e., those changes to which the individual is robust. The lemmas that follow—the demonstrations of which, in each case, are said to follow more or less directly from the definition of an individual—purport to show that an individual "will retain its nature, as before, without any change in form" in each of the following situations:

- (**L4**) certain of its parts are removed and simultaneously replaced by others of the same nature;
- (**L5**) the parts composing the individual become greater or less, but in such a proportion [*ea tamen proportione*] that they all keep the same ratio of motion and rest to each other as before [*ut omnes eandem ut antea ad invicem motus et quietis rationem servent*];
- (**L6**) certain bodies composing the individual are compelled to change the direction of their motion from one direction to another, but so that they can continue their motions and communicate them to each other in the same ratio as before [*invicem eadem qua antea ratione communicare*];
- (**L7**) whenever the individual *as a whole* [*id secundum totum*] moves or rest, or moves in this or that direction, so long as each of its parts retains its motion, and communicates this motion, as before, to the other parts.

By locating the 'oneness' of individual bodies in that which is invariant or fixed in the particular patterns or ways according to which motions are *communicated* between the parts, individual bodies can develop robustness to many changes at various levels (meaning, can change in certain ways without ceasing to be the 'individual' they are). What is curious is that, for all Spinoza's emphasis on this concept, and for all the discussion in the secondary literature of Spinoza's treatment of motion and rest and his "physics," very few commentators seem to make a vital and rather basic distinction. Most commentators, in referring to this definition of *one body*, go on to speak of individual bodies as distinguished by a "fixed ratio of motion and rest." It is true that very occasionally, such as in Lemma 5 above, Spinoza himself invites this formulation.[2] But, more generally, and in particular in his definition of what makes a composite body an individual and in the sixth and seventh lemmas, this "fixed ratio of motion and rest" notion is importantly *not* what Spinoza himself says. Rather, he states that the parts or "number of bodies" of a composite body will form an individual or *one body* whenever those parts *communicate* their motions to each other in a certain fixed proportion/ratio/pattern [*motus suos invicem certa quadam ratione communicent*].[3]

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[2]But note that lemma 5 comes in the context of specifying what sorts of changes to its parts an individual can endure without ceasing to be that individual. Any composite of bodies, the parts of which communicate their motions to one another in a certain fixed pattern will be an individual, an individual that, *by virtue of* the invariant manner in which its parts communicate their motions to one another, can also be spoken of as having parts that maintain a certain fixed ratio of motion and rest. But I think it cannot be true that if parts of a body have a certain fixed ratio of motion of rest, then they must communicate their motions to one another in the same way. One might thus best think of the "fixed ratio of motion and rest" phrase as something of a "shorthand" for the more fundamental "communication" account; but, conceptually, the two should be distinguished, as the latter is more fundamental.

[3]That commentators in general tend to ignore or elide this component of 'communication' is perhaps partially due to the fact that in Shirley's influential translation, he had elected to simply remove the word 'communicate' entirely, resulting in a very loose and misleading translation of the definition: "When a number of bodies of the same or different magnitude form close contact with one another through the pressure of other bodies upon them, or if they are moving at the same or different rates of speed *so as to preserve an*

While Descartes's notion of "transference" was meant to cover "simple encounters" (e.g., collisions) between bodies that are touching or contiguous, Spinoza's notion of "communication," because it is meant ultimately to apply to "higher-order" interactions between (highly) composite bodies, is not restricted to contact-based transmissions of motions. The power of the notion of communication is that, unlike the notion of transference (which is necessarily bound to "closeness"), it "scales" well. While the real force of the concept of communication comes in its iteration at "higher and higher" (more and more composite) levels, I can briefly indicate how one might understand this notion of communication in the context of very simple interactions, referring back to our previous simple example of a motion from Chapter 4:

8	9	10
14	15	16
20	21	22

9	10	11
14	15	16
21	22	23

$T_1 \longrightarrow T_2$

While we were originally focused on the central body 15 together with its neighborhood of nearest neighbors, we saw how, on Descartes' account, 15 in fact formed a part of the

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unvarying relation of movement among themselves, these bodies are said to be united with one another and all together to form one body or individual thing [...]" (Spinoza, *Spinoza*, 253, my emphasis).

This forgetting of the "communication" aspect of the definition is reproduced, in thought, even when the translation restores the word, leading to all kinds of distracting debates about, for instance, what Spinoza really meant by a "fixed *ratio* of motion and rest," i.e., whether we should interpret *ratio motus et quietis* in a "quantitative" way (as a literal ratio) or non-quantitative/non-technical ("a pattern or proportion"), etc. While I do not see why Spinoza cannot easily intend the "ratio" here to be *both* a literal ratio and a more generic "pattern," I also believe this is less important than the fact that the interlude as a whole, with its continual emphasis on *communication* between the parts, takes the most decisive thing to be that there is a stable manner of communicating motions, and does not appear to be particularly concerned with whether or not we should put certain restrictions on the form this stable manner of communication can take. I take it that it may need some further thought on the part of the reader that, textual evidence aside, it is not the same thing to say that a body is one body whenever its parts conform to a *fixed ratio of motion and rest* as it is to say that a body is one body whenever its parts *communicate their motions to each other in a certain fixed proportion/pattern/ratio*—however we interpret the meaning of *ratio*.

'one body' 14-15-16, due to the nature of the neighborhood-transfer definition and the fact that the entire unit 14-15-16 transfers its neighborhood "together." For Spinoza, we can also observe that the latter unit will be 'one body', but now precisely because the motions that the bodies 14 (or 16) can communicate to 15 (and vice versa) in T_1 are the same as those communicable between them at T_2. By contrast, we can observe that 9 (for instance), considered in relation to 15, can communicate only downward motions to 15 (or "receive" upward motions from 15) and "diagonal" motions to (or from) 14 and 16 at T_1. But at T_2, 9 can communicate downward motions to 14 (and receive any upward motions from 14), can communicate diagonal motions with 15, and can no longer communicate directly with 16 at all (since 16 is no longer one of its nearest neighbors). This is a change in the sorts of motions that are communicated to and from the neighbors of 14-15-16; but, importantly, the motions communicable between 14, 15, and 16 remain the same through both times. A similar analysis can be performed for each of the other nearest neighbors (establishing which are to be regarded as 'one' and which move and rest independently).

In the above example, we have understood a *motion* (*rest*) in terms of the difference (from one moment to another) in what can (cannot) be communicated between a body and its nearest neighbors. But how did we define or delimit 'one body' as against its nearest neighbors? Some bodies w, y, and z (or $14, 15$, and 16) are in fact *parts* of one body (namely 14-15-16), since, *between themselves*, they can only communicate the same motions in the same way from T_1 to T_2. We can call this the "continuity in communication of motions" condition. 14-15-16 are *one body* because whatever motion(s) 14 (or 16) can communicate to 15 (and vice versa) at T_1, and whatever motions 14 can communicate to 16 (and vice versa) via 15, the exact same motions will be communicated in exactly the same way at T_2. Since we defined bodies as we have, and since we defined a motion to be the difference (from one moment to the next) in what can be communicated between a body and its nearest neighbors, we have really been saying that a motion is the difference (from one moment to another) in the differences/changes that can be communicated between a fixed pattern of communication (of differences in what can be communicated) and a pattern of communication that, with respect to its effects on the nearest neighbors of the former pattern, does not remain fixed. This may seem overly complex, but it is really hardly different from what goes on when we consider relations of relations of relations, as discussed in the previous chapter. And understanding how the definition can be iterated thus, building on itself, is a key part of understanding the sorts of metaphysical claims it will be used to support.

Having made these remarks about the importance of attending to the centrality and ineliminability of the *communication* aspect of the definition, let us return to the interlude itself. Spinoza immediately follows the seventh lemma's demonstration (which just appeals to the definition) with the remark that "By this, then, we see how a composite individual can be affected in many ways, and still preserve its nature."[4] It is worth noting that, despite the fact that there is continuity between the accounts of what holds for the simplest bodies and what is true for the composite bodies, change or changelessness will look somewhat different for an individual *composite* body than it will for the simplest bodies. For the latter are simply regarded as some definite motion or rest, something that can and will only change from *without*, through some determination by an external body. By virtue of this "principle of inertia," such bodies will refrain from changing their state only *negatively*—by virtue of

[4]Spinoza, *Ethics*, II, "Physical Interlude," L7 Schol.

not being forced to change their state by the action of another external body. In short, to the extent that such bodies can even be the cause of their own invariance (or changelessness), this is a very "uncomplicated" type of invariance, in the sense that any change whatsoever can only be a change to the body's definite state of motion or rest, and since such a body just *is* its definite state of motion or rest, any change will result in a change in the body itself. In other words, there is no way that such "simplest bodies" could ever change *in some respect*, while remaining what they are. The former composite individual, on the other hand, may endure many kinds of changes to its parts and the motions of its parts without thereby changing or giving up the 'one body' that it is. And as an individual becomes *more* composite, this robustness to certain changes will become even more complex, allowing the individual to change in even more "higher-order" ways without thereby changing as a whole. This can be seen especially clearly in Spinoza's subsequent compressed account of how his theory of composite bodies can be extended:

> So far we have conceived an individual which is composed only of bodies which are distinguished from one another only by motion and rest, speed and slowness, that is, which is composed of the simplest bodies. But if we should now conceive of another, composed of a number of individuals of a different nature, we shall find that it can be affected in a great many other ways, and still preserve its nature. For since each part of it is composed of a number of bodies, each part will therefore (by L7) be able, without any change of its nature, to move now more slowly, now more quickly, and *consequently communicate its motion more quickly or more slowly to the others.*[5]

As we pass to even more composite individuals, even the *speed* with which the motions of the parts are communicated to one another can change without thereby altering the individual's

[5]Spinoza, *Ethics*, II, "Physical Interlude," L7 Schol., my emphasis.

nature (as long as some still "higher-order" feature of the communication of motions stays the same). But we do not, of course, need to stop here, with composite bodies that are composites of a number of individuals of different natures, with its specific level of "higher-order" robustness:

> But if we should further conceive a third kind of individual, composed [of many individuals] of this second kind, we shall find that it can be affected in many other ways still, without any change of its form.[6]

We might say that, among other things, such a "third kind of individual" will be robust even to changes in *acceleration* of the communication of the motions of the parts to one another (because it would *be* some fixed pattern of communicating motions with a certain acceleration). But why should we stop there, with this third kind of individual, i.e., composites of composites that are themselves composites of individuals of different natures? According to Spinoza, it is precisely in continuing with this recursive procedure that we arrive at perhaps *the most fundamental idea of the* Ethics:

> And if we proceed in this way to infinity [i.e., conceiving composites of composites of composites of . . .], we shall easily conceive that *the whole of nature is one individual, whose parts, that is, all bodies, vary in infinite ways, without any change of the whole individual.*[7]

With this bold statement, Spinoza brings the interlude to a close. We can recall that with the interlude he intended primarily to provide a foundation for a closer examination of the "superiority" of the highly composite human body and the corresponding hierarchy of animation or among minds. But, he adds, "if it had been my intention to deal expressly with

[6]Ibid.

[7]Ibid., my emphasis.

body, I ought to have explained and demonstrated these things more fully." Instead, he concludes the interlude by recording six postulates that apply the above theory of composite bodies to the human body in particular, postulates that most notably assert how the human body is composed of a great many individuals of different natures, themselves highly composite (Postulate 1); how the individuals composing the human body—and thus, the human body itself, as a whole—are affected by external bodies in very many ways (Postulate 3); how the human body, to be preserved, requires a great many other bodies which continually regenerate it (Postulate 4); and how the human body can move and dispose external bodies in a great many ways (Postulate 6).

Through the "infinite" recursive progression of composites of composites of . . . , the level or "excellence" of an individual body on the "hierarchy" or scale of perfection is determined by the degree to which that individual is robust to being affected in more ways (i.e., by the degree to which it can be introduced to new affections while preserving its characteristic patterns of communicating motions between parts). At each higher level (of compositeness), the individuals can affect and be affected in more ways, and in more complicated ways, without changing their form or pattern of communicating motions. These different thresholds of robustness to a greater diversity of affections are what ultimately represent the characteristic degree of power of an individual body.[8] This is, again, in direct contrast to our ordinary "naive" ideas about individuality or the unity of a body, which attempt implicitly to align it with something like *simplicity* or what is non-composite. There is no such alignment

[8] "But how can that be true, since a body can be affected *passively*?!" Well, I deliberately phrased things in terms of a greater diversity of affections *together with* an invariance or robustness of the individual. I will make this clearer below, but suffice it to say that having both of these together is specifically to require that the affections are *active*.

in Spinoza's theory. This is something that is made especially evident in the propositions that follow the interlude, where Spinoza shows that the idea constituting the *esse* of the human mind—as the idea of a body that is composed of a great many highly composite individuals—is *not simple*, but composed of a great many ideas.[9]

In our many encounters, we encounter certain bodies that are compatible with our own body, that compose well with our own—the existing communication system between our body's parts can be integrated with their pattern of communicating motions, or they can integrate our body as a whole, rendering it a part of a larger individual capable of communicating in a certain fixed manner with a greater number and variety of bodies, while simultaneously preserving the manner with which our body's parts already communicate their motions among themselves. The idea is that if x and y and z form 'one body', say X, defined as 'one' by virtue of some fixed pattern P_1 of communicating motions to one another, then in being integrated (now as a part) into some other 'one body', say Q, characterized by some other fixed pattern P_2 of communicating motions, X will at once form a part of Q without this contribution requiring that X, as a whole of its own, cease to communicate motions among its own parts according to the pattern P_1. In this way, such fixed patterns can "nest." But understanding this requires that we better understand what it means for a 'whole' or 'one' to come to form a part of another 'whole', without ceasing to be its own 'whole'. And this will involve a careful look at the common notions—or, more precisely, at that, at the level of body, of which the common notions are the idea.

[9]See Spinoza, *Ethics*, II 15.

The more compatible bodies we extend our own body into, i.e., the more composites our own body becomes a part of, the fewer things there are in Nature that might threaten our body's integrity. All else being equal, if our own fixed pattern of communicating motions among our parts is preserved as we compose with other bodies, this increases our ability to affect and be affected by more bodies and in more ways, extending the reach of the number and sort of bodies with which our body can communicate its motions, increasing our resilience to disintegration and threat by other, incompatible bodies, while we remain the 'one body' we are. All these ways of retaining our characteristic robustness to certain changes while attaining a greater variety of affections produce certain *affects of joy*. We are naturally led to seek out such joyful affects, and avoid or limit the effects of the sad affects, those encounters with other bodies that make us give up some of our robustness in exchange for an introduction to new affections and power to affect others. The transitions in our power that such affects represent are, in truth, nothing more than the reflection of how well *our body is composing with other bodies*. Because our bodies are highly composite, in being affected, it might be that our body *as a whole* is affected in some way, or that only a part or some conjunction of parts is affected. The affects that affect us daily typically relate to only a part of the body which is affected more than the others; any such affect that relates not to our body *as a whole* but to some part or conjunction of several parts, affecting that part or parts more than others, is a "passive affect"—and this cuts across the distinction between joy and sadness, and includes affects of "pleasure." It is worth emphasizing that what makes for this vital distinction between active and passive affects is whether the affection is equally of a part and the whole of our body or if it affects certain parts more than others, i.e., affects part(s) at the expense of the whole. The main issue with the passive affects, and what indeed makes for their "passivity," is that the determinations of our body involved in such affects "prevent the body from being capable of being affected in a great many other ways."[10] By contrast, those ways of being affected that affect a body *as a whole*, necessarily involve its positive power of action, giving us the "active affections" of the body. While the passive affects cut across the distinction between what causes joy and what causes sadness, the active affects are necessarily joyful for they at once introduce our body to a greater variety of relations with other bodies while also ensuring—by virtue of the fact that all parts of our body are affected together ("simul") and equally—that all the parts of our body can continue to communicate their motion and rest to one another in the same fixed manner. Whenever we are affected with joyful affects, such affections thus correspond, at the level of the ideas, to the formation of images of objects that agree in nature in some way with our body. What is positive in a joyful affect gets at definite communicative capacities that are shared by the bodies involved in the encounter, capacities that reveal themselves in an affection that at once involves the other body as cause and draws on our body's own power to act as a cause. Our body goes through all sorts of changes in terms of changes to our body parts, to those bodies with which we compose, even to the way all the parts of our body are affected—yet there are features of our body as a whole (its distinctive pattern of communicating motions) that remain robust to such changes. To know the *cause* of something, in general, is to have the idea of how to change or control certain aspects of some things or their combination without affecting the rest. To know, then, which aspects of our body remain invariant through changes to our parts and various encounters with other bodies, is to know what our body contributes as a

[10] Ibid., IV 43 Dem.

cause and what is to be attributed to other bodies as causes. In general, bodies that have little in common or are far apart have much less effect on one another than those that are "close" (both in terms of shared powers or in terms of proximity).

Discovering what is not just common between our body's pattern of communicating motions and the pattern of communicating motions characteristic of another body, but that which, as our body composes with another body, is "equally in the part and in the whole" (i.e., characteristic of us 'as a whole' and of the new 'whole' of which we now form a part), is captured, at the level of ideas, in what Spinoza calls the *common notions*. As already mentioned, whenever we are affected with joyful affects, such affections correspond to the formation of images of objects that agree in nature in some way with our body. Since a common notion inherently applies to several bodies and relates images of things to more things, it is thus more frequent, flourishes more often, and more easily engages the mind.[11] Thus, in new encounters, all else being equal, our common notions seem to thrive, to "out-compete" those images that refer only to a limitation of our body. In this way, common notions determine the mind, upon being affected by the image of an object, to consider several objects; common notions reduce the intensity of passive affects and increase the liveliness and frequency of active affects. We notice, though, that the less general our common notion—the more it is the idea of some communicative capacity that is "proper to" just our body and the other body with which we are composing—this corresponds to an ability to more easily align our body's powers with those of another. On the other hand, when confronted with a body that seems incompatible with our own in a great number of ways, we are left seeking

[11] See Ibid., V 11-13.

out a commonality that, at the level of ideas, corresponds to a "more general" common notion—in the limit, if nothing else, we can at least say that it must be a body, and so share those properties *any* body whatsoever must have. This process of locating commonalities of differing degrees of generality between our own bodies and other bodies could be said to correspond to a process whereby our body discovers itself (as cause) outside itself, to varying degrees, distributed throughout Nature. The ways we develop for locating such commonalities (from the least general to the most general), and the resulting system, at the level of our ideas, comprised of common notions of differing levels of generality, usually goes by another name: reason. Common notions let us understand the *necessity* of the various levels of agreements and disagreements, compatibilities and incompatibilities, that exist between bodies.

Our efforts to use our reason—to find more commonalities and to order these—amount to efforts, at the level of our body, to place our "highly composite" body in still higher and more complex composites. Reason is a mode of organizing our ideas that parallels the process of establishing extended communities or "societies" of composition. The process of reason exactly parallels the processes whereby our body strives to embed the systems of relations that obtain between the parts of our body within larger, more powerful and more composite, systems of relations that embrace many more composite bodies and ways of being affected. When we find that, or can make it so that, our body agrees entirely (or almost entirely) with another, all (or almost all) of our relations can be combined, a combination that has the curious feature that it at once increases the affections available to our body while (on account of the fact that the composition is made via what is already shared) also ensuring that our body's parts continue to communicate their motions among themselves in the same fixed way. In addition to clarifying Spinoza's contributions to the connections between continuity and generality, we will better understand the nature of composition of bodies if we can understand what, at the level of body, corresponds to the crucial aspect of common notions as being "equally in the part and in the whole." It is thus no surprise that much of the remainder of Book II of the *Ethics*, following the "Physical Interlude," is devoted to establishing how we come to know *adequately*, via the common notions.

Returning to the flow of Book II, then, consider that the affections of our human body in particular involve both the nature of our body and that of external bodies, a fact that begins to account for why the mind does not have a distinct idea of the body itself, and so does not know itself (the mind), except through ideas of affections of the body,[12] a claim that further accounts for why the human mind in particular does not involve adequate knowledge of the parts composing the human body (IIP24). In his demonstration of this last proposition, Spinoza writes:

> The parts composing the human body pertain to the essence of the body itself only insofar as they communicate their motions to one another in a certain fixed manner (see the definition after L3C), and not insofar as they can be considered as individuals, without relation to the human body. For (by Postulate 1) the parts of the human body are highly composite individuals, whose parts (by L4) can be separated from the human body and communicate their motions (see A1″ and L3) to other bodies in another manner, while the human body completely preserves its nature and form. And so the idea, *or* knowledge, of each part will be in God (by P3), insofar as he is considered to be affected by another idea of a singular thing (by P9), a singular thing which is prior, in the order of Nature, to the part itself (by P7). The same must also be said of each part of the individual composing the human body. And so, the knowledge of each part composing the human body is in God insofar as he is affected with a great many ideas of things, and not

[12] See, Ibid., II 19, 23.

> insofar as he has only the idea of the human body, that is (by P13), the idea which constitutes the nature of the human mind. And so, by P11C, the human mind does not involve adequate knowledge of the parts composing the human body. Q.E.D.[13]

The main thing to observe here is that the parts composing the human body pertain to the *essence of the body* itself only insofar as they communicate their motions to one another in a certain fixed manner—*not* insofar as they can be considered as individuals themselves, without relation to the human body. As should be expected by now, the nature or essence of a body is not some collection of parts. In elaborating on this, Spinoza says that the parts can be "separated from the human body" and still communicate their motions to other bodies in another way, all while the human body "completely preserves its nature and form." This amounts to yet another very explicit statement that (1) the essence of a body is the invariant way in which the motions of the parts are *communicated to one another*, and (2) the element of *communication between parts* is so central here, that it can even be said that there is nothing to be found in the parts themselves, regarded in isolation, that pertains to the essence of the body of which they form a part. I am emphasizing this so that the reader can more readily appreciate how the theory of what makes for 'one body' and how bodies compose, as presented in the interlude, is not some isolated curiosity, but will form an integral part of many of Spinoza's most important chains of reasoning.

The same sort of reasoning accounts for why the idea of any affection of the human body does not involve adequate knowledge of an external body, another individual (IIP25), and why, moreover, no idea of any single or isolated affection of the human body will in-

[13] Ibid., II 24 Dem.

volve adequate knowledge of the human body itself (IIP27). All of this culminates in the propositions IIP28-29, which tell us that the ideas of the affections of the human body, insofar as they are related to the human mind alone, are not clear and distinct, but confused (IIP28), and the idea of the idea of any affection of the human body does not involve adequate knowledge of the human mind (IIP29). The corollary to P29 begins to put this all into perspective:

> From this it follows that so long as the human mind perceives things from the common order of Nature [*ex communi Naturae ordine*], it does not have an adequate, but only a confused and mutilated knowledge of itself, of its own body, and of external bodies. For the mind does not know itself except insofar as it perceives ideas of the affections of the body (by P23). But it does not perceive its own body (by P19), except through the very ideas themselves of the affections [of the body], and it is also through them alone that it perceives external bodies (by P26). And so, insofar as it has these [ideas], then neither of itself (by P29), nor of its own body (by P27), nor of external bodies (by P25) does it have an adequate knowledge, but only (by P28 and P28S) a mutilated and confused knowledge. Q.E.D.[14]

In the Scholium, Spinoza elaborates on what he means by "perceiving things from the common order of Nature," claiming that by this he means that it is "determined externally, from fortuitous encounters with things, to regard this or that," something he opposes to being "determined internally, from the fact that it regards a number of things at once [*simul*], to understand their agreements, differences, and oppositions."[15] For, he adds, "so often as it is disposed internally [*interne disponitur*], in this or another way, then it regards things

[14]Ibid., II 29 Cor.

[15]Spinoza, *Ethics*, II 29 Schol. Note that by "the common order of Nature" Spinoza appears to mean the infinite chains of causes, something that necessarily involves the duration of singular things being determined to exist and produce certain effects in a certain way by an infinite chain of causes. This "common order" is not to be confused with the *common notions*, as we shall see.

clearly and distinctly."[16] But what exactly does it mean to be "disposed internally"? Thus far, Spinoza appears to mean by this that "it regards a number of things at once or together [*simul*]." The language of taking a number of things "at once or together" strongly suggests that Spinoza is thinking very much of the correlate, at the level of a mind, of what is involved in the process of composing *one body*. Moreover, the further suggestion, introduced in the very next proposition and developed over the course of a chain of 15 interlocking propositions, that in fact, we *do* have adequate knowledge and that "those things that are common to all" can only be conceived adequately,[17] is held to depend on the idea that our body can come to be disposed in more ways *at once* or *together*, i.e., that the limits of our body can be expanded by composing with still more composite individuals. The relevant chain of reasoning, which will lead us more directly to the heart of the matter concerning common notions and their connection to the composition of bodies, can be paraphrased thus:

- P36: All ideas are in God or Nature and, insofar as they refer to God, they are adequate: "and so, there are no inadequate or confused ideas except insofar as they refer to the singular mind of someone (see P24 and P28)."[18]
- P37: "What is common to all things (on this see L2 above) and is equally in the part and in the whole, does not constitute the essence of any singular thing."[19]

[16]Ibid., II 29 Schol.

[17]See, Ibid., II 38.

[18]Ibid., II 36 Dem.

[19]Ibid., II 37. While I do not have space to discuss this distinction in detail, this is one of a number of places where it becomes clear that by 'singular thing', Spinoza cannot mean the same thing as the 'Individual' referred to in the definition of 'one body'. But see Spinoza, *Ethics*, II Def 7; 24 Dem.

- P38: "Those things which are common to all, and which are equally in the part and in the whole, can only be conceived adequately" (IIP38), from which it follows "that there are certain ideas, *or* notions, common to all people. For (by L2) all bodies agree in certain things, which (by P38) must be perceived adequately, *or* clearly and distinctly, by all."[20]

- P39: If something is common and proper to [*commune est et proprium*] the human body and certain external bodies by which the human body is usually affected [*affici solet*], and is equally in the part and in the whole of each of them, its idea will also be adequate in the mind.

> Let A be that which is common to, and proper to, the human body and certain external bodies, which is equally in the human body and in the same external bodies, and finally, which is equally in the part of each external body and in the whole. There will be an adequate idea of A in God (by P7C), both insofar as he has the idea of the human body, and insofar as he has ideas of the posited external bodies. Let it be posited now that the human body is affected by an external body through what it has in common with it, that is, by A; the idea of this affection will involve property A (by P16), and so (by P7C) the idea of this affection, insofar as it involves property A, will be adequate in God insofar as he is affected with the idea of the human body, that is (by P13), insofar as he constitutes the nature of the human mind. And so (by P11C), this idea is also adequate in the human mind. Q.E.D.
> Cor: *From this it follows that the mind is more capable of perceiving many things adequately as its body has many things in common with other bodies.*[21]

The final corollary tells us that the mind is more capable of perceiving many things—and perceiving them *adequately*—the more its body has many things in common with other

[20]Ibid., II 38 Cor.

[21]Ibid., II 39 Dem-Cor., my emphasis

bodies. I am repeating this because it is vital to realize that Spinoza has here aligned an increase in adequate knowledge with an increase in what our body has "in common" with other bodies, which tells us that in attaining a more adequate knowledge of more things, our body is establishing more commonalities with other bodies. But as the demonstration above also emphasizes, the adequate idea corresponds not just to what is equally in our body and the other body (or bodies), but to what is "equally in the part of each external body and in the whole." Another vital element of the previous chain of reasoning is how it gives rise to something I indicated earlier: namely, an implicit theory of various *levels* of common notions. On the one hand, in P39 we see that whenever something is both *common and proper to* my human body and another external body (or bodies) by which my human body is affected, and is *equally in the part and in the whole of each of them*, then its idea will be adequate in the mind. It suffices to recall the definition of 'one body' to realize that, in speaking of what is "common" to one body and another body or bodies, Spinoza means to indicate commonalities among patterns of communication of motions. In speaking of what is "proper to" the bodies involved, Spinoza is referring to the fact that what is common may be specific to the two (or more) bodies involved in the encounter, i.e., in the case of just two bodies, the commonality may be least universal or general. On the other extreme, there are common notions that capture "those things which are common to all, and which are equally in the part and in the whole, and so can only be conceived adequately"[22]—something that follows from the second lemma of the interlude, namely that all bodies agree in certain things. There is, necessarily, an entire spectrum between these two extremes, ranging from the least

[22]Ibid., II 38.

general to the most general. And in all cases, regardless of "level," it is characteristic of common notions not just that it is an idea the object of which is something common to distinct bodies, but that *it is equally in the part and in the whole.* I take it that this means, above all, that there is not just some "abstract" feature shared by the bodies, but that the underlying bodies are in fact composing with one another to form 'one body' (or that one is being subordinated to the other body's principle of unity). There are thus as many levels in the common notions as there are different 'ones' or complexes with which our body can compose—from the fact that my body might have something proper to and in common with another body, to something proper and common to many other bodies, all the way up to whatever is no longer "proper" but is common to all bodies, and which is equally in the whole (*all* of extended Nature) and in each part. The "coherence" of Nature as a whole, the fact that everything is "in" Nature as a whole, is reflected in this scale of generality. A "more general" common notion will be one that refers, at the level of body, to actual powers to affect and be affected by a greater number and diversity of affections without this diversity compromising the unity of the 'one body'.

This all leads to P45, which tells us that each idea of each body—of each singular thing that actually exists—necessarily involves an "eternal and infinite essence of God." In the demonstration of the subsequent proposition, which tells us that the knowledge of God's eternal and infinite essence which each idea involves is adequate, Spinoza claims that

> whether the thing [i.e., each body] is considered as a part or as a whole, its idea, whether of the whole or of a part (by P45), will involve God's eternal and infinite essence. So what gives knowledge of an eternal and infinite essence of God is common to all, and is equally in the part

> and in the whole. And so (by P38) this knowledge will be adequate. Q.E.D.[23]

The most important thing to realize in all this is that just as *communication* was the key to understanding when a composite body is *one body*, the fact that something common to various bodies is *equally in the part and in the whole* is the key to understanding the *common notions*. The fact that a body is a certain fixed pattern by which the parts *communicate* their motions to one another is the basis of the fact that there are *common notions*, where the latter capture whatever is *both* common to two or more bodies *and* equally in the parts of each body and in the whole now formed by their composite body. When one body, as a fixed manner of communicating motions between its parts, not only has something in common with another body (as another fixed manner of communicating motions between its parts), but if the bodies come to compose a new 'one body' along the lines of this commonality in (a portion of) their original manner of communicating motions, and do so in such a way that the common pattern of motion remains equally in each individual body but now also appears in the new 'whole' or 'one' of which they compose parts, then, at the level of ideas, this process is captured by a common notion. This same sort of structure of composition happening at the level of the object of which the common notion is the idea is also arguably reflected, as "in parallel," at the level of ideas, in the relations between ideas represented by common notions—these are ideas that are highly composite but that capture, in the form of one idea, an invariant way many ideas communicate with one another. Common notions are

[23]Ibid., II 46 Dem.

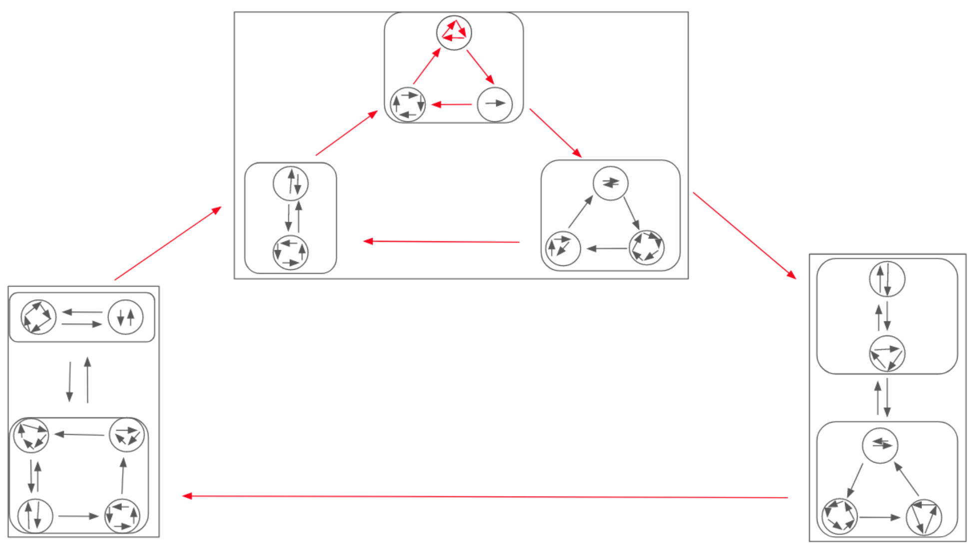
thus highly reticulated 'unities', and their power lies in their ability to bring together and relate, under a single principle of unity, many other component ideas.[24]

When, in the previous chapter, I spoke of the recursive nature of the paired definition of one body and motion in Cartesian physics, and I said that the same sort of recursive structure would reappear in Spinoza's definition, I meant also to anticipate this aspect of *self-similarity* characteristic of that which corresponds, at the level of body, to the common notions. This self-similarity can be observed in two "registers": a pattern of communication of motions that obtains between the parts of some 'whole' can be observed, as that body composes with other bodies to form now a part of another 'whole', throughout the new whole; or new 'wholes' are composed of copies of itself 'glued together' in some fashion, i.e., 'wholes' are recursively composed.

By "self-similarity," I mean above all to indicate the particular aspect of common notions as corresponding to not just what is common to two or more bodies but what is *equally in the parts and in the whole*. Let me say a few more things about this. Assume there was no entity or collection of entities X for which the way the parts of X communicate their motions among themselves was "copied" (downward) at the level of one of the parts themselves (among each of their own parts) or (upward) at the level of another entity of which X formed a part. In such a world—*especially* if we accept that what a body *is*, as an individual, is a fixed manner of communicating motions between its parts—we would never be able to predict the effect of combining or composing several bodies (and their characteristic

[24] I think this is one particularly fruitful way to understand Spinoza's claims, presented in the context of discussing the common notions, concerning "how a person can know that he has an idea which agrees with its object," something that "arises solely from his having an idea which does agree with its object—*or* that truth is its own standard" (Spinoza, *Ethics*, II 43 Schol.).

actions) from knowing only the effects on each of the parts. Think of it this way. Suppose you are tasked with building a universe. You start with some "simplest bodies," entities that are characterized by their motion or rest alone. You realize that these motions and rests can change, but that on their own, such changes merely change *which things* are moving and *which things* are at rest, which does not accomplish much in the way of achieving any sort of more involved organization. You do not want to have to introduce any new *type* of entity, but you would like your universe to be a little less "boring," so you try to relate these entities in some fashion so as to assemble more complicated entities. You realize that these entities already interact, and interact to change one another's motions, certain entities already communicating their motions to one another but not to others. Moreover, of these ways of communicating motions, some such communications appear to have achieved some form of stability, appear to have assembled into something like a predictable pattern of communication capable of continually "digesting" new entities in motion, and relaying those motions to other entities in a certain fixed manner. This is exciting, for it seems to have saved you from having to invent any *new kind of entity.* There are still just the motions and rests with which you began, but now the universe is beginning to *organize* itself somewhat, to take shape. At this point, you begin to wonder if you might not take these local systems of fixed patterns of communication of motions and assemble them into larger, even more complex systems. It strikes you as a good idea to continue to work with what you already have, not introducing anything new, yet somehow producing greater complexity in the interactions. You consider that one way to meet this condition in a particular elegant way would be to link together distinct patterns of communication of motions according to the model provided by one of the patterns of communications already found in one of the bodies you will use as a part to assemble a greater whole. At each new "level," by making the new pattern of communication structurally similar to the pattern of communication found in certain of its parts, you ensure that, while introducing ever more complex forms of organization, there remains a sort of continuity. This suddenly strikes you as pretty clever, for you immediately realize that, proceeding in this manner over and over again, there is no upper limit to the complexity or organization you can introduce, without ever introducing any new kind or type of entity, and while ensuring a kind of "sympathy" between changes to certain relationships among parts and changes to the whole of which they form a part. At a very abstract level, a picture of how you compose "higher-order" individuals through a process of iteration, ensuring a kind of self-similarity, might look something like this:



The idea I mean to convey with this (very rough) image is that the sorts of relations at one "level"—where arrows abstractly represent relations, and the "shape" of the arrow complexes abstractly represents something like the *form* of the system of relations—reappear at other levels. For instance, one can see the complex of red arrows, taking on the "shape" of a triangle, reduplicated at four different levels.

We know, from the definition of an individual or one body, that it makes sense to conceive of a part of something as itself a "thing" or individual only where there exists a certain coherency to the relationships or patterns of communication among the parts of that part. What would happen if our universe were such that whenever it made sense to think of a part as a 'one' in its own regard, the coherency found in the relationships or patterns of communication between the parts of that part was never "duplicated" at the level of any "higher" whole's pattern of communication? For one thing, in such a universe, I already suggested that we could not expect to be able to predict the effect on any whole from knowledge of the effect on its parts. Moreover, we would not expect to be able to deduce anything about the coherency of a part from that of our knowledge of the coherency or 'unity' of a whole. The idea of the coherency determining a part as 'one body' in its own regard would not involve the idea of the coherency of the whole, and vice versa. As it turns out, our universe is not like this. But, perhaps more importantly, if such a universe were to exist, it is not obvious that we could ever change or control some part of an entity without affecting all the rest. Moreover, it is fairly clear that, in such a universe, there could be no invariance in the universe as a whole. Finally, in such a universe, it is certainly unclear that we could ever expect to say *anything* meaningful about the concept of Nature *as a whole*.

On the other hand, suppose our universe was a universe where at least sometimes, for certain bodies and in certain respects, we expected to be able to predict the effect on the whole in some interaction from knowledge of the effect on certain of the parts, and we expected to be able to deduce something about the coherency of a part from that of the whole. This would be a universe that, in contrast to the first universe, would have to have some sort of structural "duplication" or "self-similarity," i.e., some commonality between the coherency found in the relationships between the parts of a part and the coherency found at the level of the various 'wholes' of which it formed a part. In such a universe, certain things could change while others stay the same. But it would also be a universe in which we could expect to come to know 'wholes' from knowledge of parts, i.e., where in particular we could be confident that in speaking of "Nature as a whole," we knew of that whereof we spoke. We could sum up this "vision" of *our* universe—a vision that joins the continuous and the general—in a slogan:

Continuity plus Self-Similar Composition equals Generality (Common Notions)

Common Notions Revisited

I would like to add a few more observations about the common notions. The common notions are distinguished by Spinoza from both *transcendentals* (such as 'being' or 'thing') and *universals* (such as genera and species, or "humanity"). Yet common notions of course are general—they have a degree of generality that depends ultimately on how that of which they are the idea is something common and "proper to" two bodies, several, or all (and everything in between). As Deleuze rightly emphasizes, the common notions are thus meant not to attack universality or all natural types altogether, but rather to attack "a certain conception of abstract universality. . . and a certain abstract determination of genera and species."[25] Deleuze goes on to write that Spinoza is here attacking not just "common sense"

[25]Deleuze, *Expressionism in Philosophy*, 277.

ideas, but specifically the Aristotelian tradition of "defining genera and species through differences":[26]

> Against this tradition Spinoza proposes a grand principle: to consider structures, rather than sensible forms or functions. But what is the meaning of "structure"? It is a system of relations between the parts of a body[...]. By inquiring how these relations vary from one body to another, we have a way of directly determining the resemblances between two bodies, however disparate they may be. The form and function of an organ in a given animal depend solely on the relations between its organic parts[...]. In the limit Nature as a whole is a single Animal[...]. For the examination of sensible differences is substituted an examination of intelligible similarities, which allow us to understand resemblances and differences between bodies "from the inside."[...] Common notions are *general* rather than *abstract* ideas.[27]

While the way our bodies are constituted requires that we first acquire common notions and begin with those that are least universal (or most "local"), and while nearly all of the *Ethics* (up to VP21) is written from the perspective of the common notions, there is another, third kind of knowing, to which the second kind allegedly leads us. This kind of knowing casts further doubt on any temptation to think that the most universal (if by that we mean "abstractly" universal) ideas can best express God or the idea of Nature as a whole. It undermines any notion that the system of reason, comprised of common notions of differing degrees of generality, tends to prioritize or culminate in the most general or universal of common notions. The measure of the power of a general idea is not first of all given by how "universal" it is, but rather by how well it combines and composes its component ideas, subordinating them to a certain principle of unity or fixed manner by which the component

[26] I hope it is clear to the reader, incidentally, that what Deleuze here refers to as the "Aristotelian tradition" does refer to a real tradition of appropriating Aristotle's thoughts on universality, but that I believe this cannot represent Aristotle's most decisive or nuanced ideas concerning the nature of genera.

[27] Deleuze, *Expressionism in Philosophy*, 278.

ideas "communicate their motions among one another."[28] The distinction between the second and the third kind of knowing comes down to this: with the second kind of knowing, we know adequately, but we do not yet necessarily have an adequate idea of ourselves as cause of this idea. Common notions are indeed explained by the essence of our body, but they certainly do not themselves constitute an idea of our particular essence. The second kind of knowing reveals in general terms that everything depends on, and is "in," God or Nature as a whole, while the third kind of knowing allows us to understand how this dependence works in the case of a particular essence. In the third kind of knowing, the knowledge still refers to what is "common" but the notion of what is "common" is transformed because there is now a kind of reflexivity to this commonality: what is common to the idea of an object and the object itself is itself what is expressed in the idea—namely, a intrinsic communicative pattern whereby the components (equally of the idea or the body of which it is the idea) communicate among themselves in a certain fixed way.

A final comment about the common notions in relation to the part-whole language. In a letter to Oldenburg (32), Spinoza famously addresses some of the reasons for being persuaded that "each part of Nature agrees with its whole, and coheres with all other parts." Spinoza points out that, naturally, he cannot say how all parts in particular cohere with others and how all really agree with the whole, since this would require "knowing the whole of Nature and all of its parts." In other words, it would require producing every last one of the common notions—from those that name what is proper to each given pair of bodies all

[28]The reader might resist this formulation, but parallelism practically demands it. The quotation marks, in that sentence, are meant to accommodate the fact that the order and connection of ideas (and so their "communications")—while necessarily "one and the same as the order of things" (IIP7)—will also look somewhat different from how this looks at the level of body (at least since ideas are not themselves bodies).

the way up to those that name what every last body has in common. However, he adds that he can supply his reasoning for being persuaded of the truth in general:

> By the coherence of parts, then, I understand nothing but that the laws or the nature of the one part adapts itself to the laws or the nature of the other part so that they are opposed to each other as little as possible. Concerning whole and parts, I consider things as parts of some whole to the extent that the nature of the one adapts itself to that of the other so that they [all] agree with one another as far as possible. But insofar as they disagree with one another, to that extent each forms in our Mind an idea distinct from the others, and therefore it is considered as a whole and not as a part.
>
> For example, when the motions of the particles of lymph, chyle, etc., so adapt themselves to one another, in relation to their size and shape, that they completely agree with one another, and they all constitute one fluid together, to that extent only the chyle, lymph, etc., are considered as parts of the blood. But insofar as we conceive the particles of lymph, by reason of their shape and motion, to differ from the particles of chyle, to that extent we consider them as a whole and not as a part.
>
> Let us feign now, if you please, that there is a little worm living in the blood which is capable of distinguishing by sight the particles of the blood, of lymph, [of chyle], etc., and capable of observing by reason how each particle, when it encounters another, either bounces back, or communicates a part of its motion, etc. Indeed, it would live in this blood as we do in this part of the universe, and would consider each particle of the blood as a whole, not as a part. It could not know how all the parts of the blood are regulated by the universal nature of the blood, and compelled to adapt themselves to one another, as the universal nature of the blood requires, so that they agree with one another in a definite way.
>
> For if we should feign that there are no causes outside the blood which would communicate new motions to the blood, and no space outside the blood, nor any other bodies to which the particles of blood could transfer their motion, it is certain that the blood would always remain in the same state, and its particles would undergo no variations other than those which can be conceived from the given relation of the motion of the blood to the lymph, chyle, etc. Thus the blood would always have to be considered as a whole and not as a part. But because there are a great many other causes which regulate the laws of the nature of the blood in a definite way, and which in turn are regulated by the blood, the result is that other motions and other variations arise in [the particles of] the blood which follow not simply from the relation of the

> motion of its parts to one another, but from the relation of the motion of the blood [as a whole] and of its external causes to one another. In this way the blood has the nature of a part and not of a whole. [. . .]
>
> Now all bodies in nature can and must be conceived as we have here conceived the blood, for all bodies are surrounded by others, and are determined by one another to existing and producing an effect in a fixed and determinate way, the same ratio of motion to rest always being preserved in all of them at once, [that is, in the whole universe]. From this it follows that every body, insofar as it exists modified in a definite way, must be considered as a part of the whole universe, must agree with its whole and must cohere with the remaining bodies.[29]

This letter makes it clear that the supreme danger—from which the common notions save us—is mistaking what is only a part (communicating motions to other parts in a fixed way so as to form a part of a greater whole) for a whole. The common notions are the way the "little worm" can come to see its way out of the bloodstream. But this is a process that requires that the little worm recognize that "there are causes external to the blood, which could communicate new motions to it...[and] bodies whereto the particles of the blood could communicate their motion." In other words, this is a process that requires that the worm consider how "other motions and other relations arise in the blood, springing not from the mutual relations of its parts only [i.e., not from the blood viewed as an Individual], but from the mutual relations between the blood as a whole and external causes," letting the worm now regard the blood as a part of another individual, not as an individual or whole itself. And this not only brings us full circle, allowing us to understand the true significance and power of Descartes' original definition (which established a non-relative way of determining what is to count as 'one body'), but also sums up the seemingly paradoxical ethical injunction implicit in the common notions as the "foundation of our reasoning": to come to regard oneself

[29]Spinoza, "Letter 32 (OP) to Oldenburg," in Spinoza, *The Collected Works of Spinoza, Volume II*, 18-19.

at the "right level," to discover the *one body* that one is and is determined "intrinsically" to be, one must first allow that the limits that appear to constitute one's own body are wider, further out than imagined. To state it somewhat polemically, living "ethically" is a matter of composing "better" with more of Nature, something that amounts, at the level of understanding, to realizations of the ways in which the 'one' that we already are is continuous with still other, more composite 'ones'.

A question that lingers, however, as far as "ethics" is concerned, can be briefly discussed with an example. Mitochondria are thought to have once been free-living bacteria. But over many years of evolution, they became more and more specialized and now they cannot live outside the cell. Certainly, from the perspective of cells, mitochondria are absolutely vital in producing the chemical energy cells need to survive, and they can be found in the cells of every complex organism. However, in having been "subordinated" to the principle of unity of a cell, in having "composed with" an individual of greater compositeness, what was once free-living bacteria appears to have "lost" something of its original individuality, of the sorts of motions communicable among its parts as well as to and from other bodies. It is not clear if this is something the individual free-living bacteria can have "strove for." There is always the danger, in other words, that in composing with more composite 'ones', our body's individuality, its capacity to function *as a whole*, will be completely transformed by its integration, *as a part*, into a "higher" individual. It is not entirely clear how, if we are that free-living bacteria, we are to evaluate the merits of what we are striving for, or if the physics of composition can tell us something *unequivocal* about the ethics of such a situation.

The next chapter largely leaves philosophy behind and introduces some profound modifications to the concept of continuity (and its connections with generality) to have emerged in one particular field of mathematics: sheaf theory. However, as will be seen, a number of connections to previous chapters will be established along the way.

Chapter 6

Sheaf Theory

Introduction

In many cases, data is localized and distributed in some fashion. In saying that the data is *local*, we just mean that it holds only within, or is only defined over, a certain region, i.e., whose validity is crucially restricted to a particular region or domain or reference context. For now, we can think of this in terms of the *partiality* of certain information, or in terms of our data being "based," i.e., as "sitting over" or being indexed by some given space or domain of sensors. We collect temperature readings and thus form a notion of ranges of possible temperatures over certain geographical regions; we record the fluctuating stockpile of products in a factory over certain business cycles; we gather testimonies or accounts about particular events understood to have unfolded over a certain region of space-time; we build up a collection of possible test results over various parts of the human body; we entertain the possible chess moves or future game trajectories traced over a chessboard; we accumulate observations or images of certain patches of the sky or the earth; we amass collections of memories or recordings of our distinct interpretations of a certain score of music; we accumulate ethical judgments or propositions about how to act in certain situations involving networks of human actors; we form a concept of our kitchen table via various observations and encounters, assigning certain attributes to those regions of space-time delimiting our various encounters with the table. Even if certain phenomena are not intrinsically local, frequently its measurement or the method of data collection may still be local.

But even the least scrupulous of humans do not merely accumulate or amass local or partial data points. From an early age, we try to understand the various modes of *connections* and *cooperations* between the data, to patch these partial pieces together into a larger whole whenever possible, to resolve inconsistencies among the various pieces, and go on to build coherent and more global visions out of what may have only been given to us in pieces. As informed citizens or as scientists, we look at the data given to us on arctic sea-ice melting rates, on temperature changes in certain regions, on concentrations of greenhouse gases at various latitudes and various ocean depths, etc., and we build a more global vision of the changes to our entire planet on the basis of the connections and feedbacks between these various data. As investigators of a crime, we must "piece together" a complete and consistent account of the events from the partial accounts of various witnesses. As doctors, we must infer a diagnosis and a plan of action from the various individual test results concerning the parts of a patient's body. We take our many observations over isolated situations involving engagements with other individuals and try to form global ethical guidelines or principles to guide us in further encounters.

Yet sometimes information is simply not local in nature. Roughly, one might think of such non-locality in terms of how certain attributes might appear to us as perceivers over a part of a space but may cease to manifest themselves over subparts of that space, in which case one cannot really think of the perception as being built up from local pieces. In the game of Scrabble™, one assigns letters to a grid of squares, one by one. One might thus suspect that we have a "local assignment" of data to a space; yet this assignment of letters to squares in order to form words is not really local in the relevant sense, since the smallest unit is really a *legal word*, but not all subwords or parts of words are themselves words—so one cannot really think of the *words* in Scrabble™as being built up from local pieces.

Moreover, even when information is local, there are many instances where we cannot synthesize our partial perspectives into a more global perspective or conclusion. As investigators, we might fail to form a coherent version of events because the testimonies of the witnesses cannot be made to agree with what other data or evidence tells us regarding certain key events. As musicians, we might fail to produce a compelling performance of a score because we have yet to figure out how to take what is best in each of our "trial" interpretations of certain sections and splice them together into a coherent single performance or recording of the entire piece. A doctor who receives conflicting information from certain test results or testimony from the patient that conflicts with the test results will have difficulty making a diagnosis. In explaining the game of rock-paper-scissors to children, we tell them that rock beats scissors, scissors beats paper, and paper beats rock, but we cannot tell the child how to win *all the time*, i.e., we cannot answer their pleas to provide them with a global recipe for winning this game.

For distinct reasons, differing in the gravity of the obstacle they represent, we cannot always "lift" what is local or partial up to a global value assignment or solution. A problem may have a number of viable and interesting local solutions but still fail to have even a single global solution. When we do not have the "full story," we might make faulty inferences. Ethicists might struggle with the fact that it is not always obvious how to pass from the instantiations or particular variations of a seemingly locally valid prescription, valid or binding for a subset of agents, to a more global principle, valid for a greater portion of the underlying network. In the case of the doctor attempting to make a diagnosis from conflicting data, it may simply be a matter of either collecting more data, or perhaps resolving certain inconsistencies by "throwing out" certain test results, i.e., ignoring certain data in deference to other data. Other times, as in the case of rock-paper-scissors, there is simply nothing to be done to overcome the failed passage from the given local ranking functions to a global ranking function, for the latter simply does not exist. The intellectually honest person will eventually want to know if their failure to lift the local to the global is due to the inherent particularity or contextuality of the phenomena being observed or whether it is simply a matter of their own inabilities to reconcile inconsistencies or repair discrepancies in data-collecting methods so as to patch together a more global vision out of these parts.

Sheaf theory is the roughly 70 year old collection of concepts and tools designed by mathematicians to tame and precisely comprehend problems with a structure exactly like the sorts of situations introduced above. The reader will have hopefully noticed a pattern in the various situations discussed above. We produce or collect assignments of data throughout certain regions. In most cases, these observations or data assignments come already distributed in some way over the given network formed by the various regions; but if not, they may become so over time, as we accumulate and compare more local or partial observations. In certain cases, together with the given value assignments and a natural way of decomposing the underlying space, there may emerge ways of restricting assignments to certain subregions of the given regions. In such cases, in this movement of decomposition and restriction, the glue or system of translations binding the various data together or permitting some sort of transit between the partial data items becomes explicit; in this way, an internal consistency among the parts may emerge, enabling the controlled gluing or binding together of the local data into an integrated whole that now specifies a solution or system of assignments over a larger region embracing all of those subregions. Such structures of coherence emerging among the partial patches or locally-indexed data, once explicitly acknowledged and developed, may enable a unique *global* observation or solution, i.e., an observation that no longer refers merely to yet another local region but now extends over and embraces all of the regions at once; as such, it may even enable predictions concerning missing data or at least enable principled comparisons between the various given groups of data. Sheaves provide us with a powerful tool for precisely comprehending the sort of local-global passages indicated above. Whenever such a local-global passage is possible, the resulting global observations make transparent the forces of coherence between the local data points by exhibiting to us the principled connections and translation formulas between the partial information, making explicit the glue by which such partial and distinct clumps of data can be "fused" together, and highlighting the qualities of the distribution of data. We may even go on to consider systematic passages or translations between distinct such local-to-global systems of data.

On the other hand, when faced with *obstructions* to such a local-global passage, we typically revise our basic assumptions, or perhaps the entire structure of our data, or maybe just our manner of assigning the data to our regions. We are usually motivated to do this in order to allow precisely such a global passage to come into view. When we can satisfy ourselves that nothing can be done to overcome these obstructions, we examine what the failure to pass from such local observations to the global in this instance can tell us about the phenomena at hand. *Sheaf cohomology* is a tool used for capturing and revealing precisely obstructions of this sort.

The main premise of this chapter is that, in a uniquely forceful way, sheaf theory enables us to better clarify the still poorly-understood connections between the phenomenon of *generality* on the one hand and *continuity* on the other. The problem is old and is one that should be very familiar. Roughly, it has to do with how generality first emerges out of the principled binding together of partial or local information in such a way that these parts and the modes of transit and action supported between such parts are coordinated, by virtue of some rule or principle or system of mutual constraints, into a coherent whole. It is closely related to the ancient philosophical problem of how we get from the 'many' to the 'one'. The major aim of this chapter is to present, in abbreviated form, something like a "tour" of the main contributions of sheaf theory to the specification of the concept of continuity (and to consider this with particular regard for its connections to the phenomenon of generality). It might also serve as an accessible initial introduction to sheaves for non-mathematicians and philosophers.[1]

We have come a long way since Chapter 1 and 2's discussions of the Aristotelian notion of continuity and its connections with a certain vision of generality. While the starting points and destinations are often quite distinct from what can be found in philosophy's development of the concept of continuity, it is curious that in the (relatively recent) history of mathematics there is a somewhat similar gradual "weaning" off of models and formulations of continuity in terms of "closeness" towards more and more "structural" and "morphological" accounts of continuity, after which there are some efforts to partially reconcile the two approaches—preserving what is best in our more "intuitive" understandings of continuity

[1]Readers with more interest in the mathematics, and who desire a much more complete and detailed introduction, are invited to have a look at my monograph on sheaves, *Sheaf Theory through Examples*.

(as fundamentally a matter of the behavior of "close things," or local interactions), while benefiting from the far-reaching aspects of the more "structural" and "morphological" formulations. Sheaf theory in particular embodies such a "reconciliation." In the conclusion of this dissertation, I will have a chance to discuss a few other models of continuity to have appeared in the history of mathematics, in relation to which some of the particular advances of sheaf theory (regarding the continuity-generality connection and the formulation of the concept of continuity) will be seen in a larger context, and some of the parallels (and differences) between the transformations the traditions of philosophy and mathematics have separately brought to the concept of continuity may then become somewhat clearer. I have elected to focus in this chapter on sheaf theory's contributions to the continuity-generality connection because I take its transformations to be one of the most powerful and nuanced of the many different contributions the history of mathematics has brought to these matters.

The rest of the chapter is structured as follows. Section 2 is dedicated to developing the concept of *presheaves* and to isolating some of the initial impacts of this concept on the notion of continuity. Section 3, the main section of the chapter, introduces sheaves, briefly touches on other matters like *sheaf cohomology*, and draws out some of the main philosophical contributions of the sheaf concept with respect to the connection between continuity and generality. The penultimate section, Section 4, builds on the previous two and considers some further aspects of the notion of a *topos*, as well as *geometric morphisms* (special sorts of maps between toposes), before building up to the notion of *cohesive toposes*. The principal purpose of this section is to discuss, in the context of cohesive toposes, how one might think more precisely of the relation between continuity and discreteness as a "dialectical" relation. I also briefly discuss to what extent we are really even dealing with a "dialectic." The final section provides a table summarizing the main philosophical results regarding the continuity-generality connection to have emerged in the course of this chapter.

Throughout each section of this chapter, I pause to highlight, in a more "philosophical" fashion, what I consider to be the most important observations or determinations regarding the concept of continuity and generality to have emerged from the preceding technical developments, drawing out at each of the "passes" the new "shades" given to the concept of continuity. A secondary aim of these "philosophical passes" sections is to periodically step back from the technical details and reflect on some of the important broader implications of the concepts or ideas underlying the technical developments. In calling these "passes," I mean of course to suggest that we will be approaching the notion various times, from different angles, and with increasing sophistication. However, I also intend to evoke the notion of a mountain pass, i.e., a route or high vantage point ideally positioned between two peaks and as such useful for increasing communication, exchange, and trade between whatever lies on both sides of the pass.

While skipping the more mathematical sections would make it rather difficult to really appreciate any of the force behind the claims and observations made, or orient oneself in the issues discussed, during the "passes" sections, those sections frequently distill much of the technical details, or at least present a very "high-level" view of those results of the preceding sections with the greatest "philosophical" import. So the staunchly philosophically-minded reader who becomes exhausted by the mathematical niceties should not feel too poorly for skimming or "lightly reading" the more mathematically heavy sections, without worrying over the fine details, instead focusing on the more descriptive claims made in those sections and the more intuitive examples (such as those provided in the "Initial Examples" section).

Such a reader is encouraged to then return with full attention to the "philosophical passes" and search back through the other, more mathematical, sections for clarification as needed.

Section 2: Prelude to Sheaves: Presheaves

The language of category theory is indispensable to the precise statement and understanding of the notions of sheaf theory. In case the reader is not already familiar with at least the basics of category theory, the next page or two briefly cover the basic idea of a category, its definition, and offers a chance for us to fix some notation. Fundamentally, the specification of a category involves two main components: establishing some *data* or givens, and then ensuring that this data conforms to two simple axioms or laws. To define, or verify, that one has a category, one should first make sure the right data is present. This first main step of establishing the data of a category really involves doing four things. First of all, this means identifying a collection of *objects*. Especially when one is assembling a category out of already established mathematical materials, these objects will typically already go by another name, like vertices, sets, vector spaces, topological spaces, types, various algebras or structured sets, and so on. Second, one must assemble or specify a collection of "morphisms" or mappings, which is just some principled way of establishing connections or relations between the objects of the first step. When dealing with already established structures, again, these will usually already have a name, like arrows or edges, functions, linear transformations, continuous maps, terms, homomorphisms or structure-preserving maps, and so on. Many of the categories one meets in practice have sets with some structure attached to them for objects and (the corresponding) "structure-preserving" mappings or connections between those sets for morphisms. Third, and perhaps most importantly, one must specify an appropriate notion of *composition* for these mappings, where for the moment this can be thought of in terms of specifying an operation that enables us to form a "composite" map that goes directly from object A to C whenever there is a mapping from A to B juxtaposed with a mapping from B to C. This composition operation in fact already determines the fourth requirement: that for each object, there is assigned a unique "identity" morphism which starts out from that object and return to itself. These four constituents—objects, morphisms, composites, and identities—supply us with the data of the category.

Next, one must show that the data given above conforms to two very "natural" laws or axioms. First, if we have a morphism from one "source" object to another "target" object, then following that morphism with the identity morphism on the "target" object should be the same thing as "just" traveling along the original morphism; and the same should be true if we first travel along the identity morphism on the source object and then apply the morphism from source to target. In short, the identity morphisms cannot do anything to change other morphisms—in this sense, they can be thought of as the "do nothing" or "degenerate" morphisms. Finally, a category must satisfy what is called the associative law, where this is just what you might expect: if you have a string of morphisms from A to B and from B to C and from C to D, then it should make no difference whether you choose to first go directly from A to C (using the composite map that we have by virtue of the third step in the data construction) followed by the map from C to D, or if you go from A to B and then go directly from B to D (using the composite map).

An entity that has all the data specified above, data that in turn conforms to the two laws described in the preceding paragraph, assembles into a category. The informal description given in the preceding paragraphs is summarized more properly in the following:

Definition 6.0.1. A *category* $\mathbf{C}$ consists of the following data:[2]

- A collection $Ob(\mathbf{C})$, whose elements are **objects**;
- For every pair of objects $x, y \in Ob(\mathbf{C})$, a collection $Hom_{\mathbf{C}}(x, y)$ (or just $\mathbf{C}(x, y)$) of **morphisms** from x to y;
- To each object $x \in Ob(\mathbf{C})$ is assigned a specified **identity morphism** on x, denoted $\mathrm{id}_x \in Hom_{\mathbf{C}}(x, x)$;
- For every three objects $x, y, z \in Ob(\mathbf{C})$, a function

$$\circ : \mathbf{C}(y, z) \times \mathbf{C}(x, y) \to \mathbf{C}(x, z),$$

called the **composition formula**, which acts on elements to assign, to any morphism $f : x \to y$ and any $g : y \to z$, the composite morphism $g \circ f : x \to z$:

$$\begin{aligned} \circ : \mathbf{C}(y, z) \times \mathbf{C}(x, y) &\to \mathbf{C}(x, z) \\ \circ(\quad g \quad , \quad f) &\mapsto (g \circ f) \end{aligned}$$

This data gives us a category provided it further satisfies the following two axioms:

- **Associativity** (of composition): if $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h} w$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] & y \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & z \arrow[r, "h"] & w \end{tikzcd}$$

- **Identity**: if $f : x \to y$, then $f = f \circ id_x$ and $f = id_y \circ f$.

[2]Throughout this document, categories are designated with bold font.

Example 6.0.1. The category **Set** consisting of sets for objects and functions for morphisms is in fact a category. (Set theory, which the reader is no doubt already familiar with, can be thought of as "zero-dimensional" category theory.)

Example 6.0.2. A group is already a category in which there is just one object and in which every morphism is an isomorphism. We can run this again to get the category **Grp** having groups for objects and group homomorphisms for morphisms.

There are many, many more examples of categories (and more "exciting" ones), some of which we will see in the following.

When Eilenberg and MacLane first defined categories, functors, and natural transformations in 1945, they stressed how it provided "opportunities for the comparison of constructions [...] in different branches of mathematics," but with Grothendieck's Tohoku paper a decade later, it became clear that category theory was not just some convenient way of comparing different mathematical structures, but was *itself* a mathematical structure with significant intrinsic interest. In other words, we do not just have categories consisting *of* mathematical objects/structures, but equally important are categories that allow us to view categories themselves *as* mathematical objects/structures. For our purposes, however, the real power of category theory is perhaps most conspicuously on display in how, by putting everything on the same "plane," we can consider principled relations *between* categories.

It is often said that category theory privileges relations over objects (or, at the least, that they are on equal footing with objects). The behavior of objects in relation to other objects (something that is given by morphisms) is at least as relevant as the structure of the objects themselves. But a category itself can be considered as an object, and then a natural question is 'what do relations between categories look like?' If a category is a context for studying a specific type of mathematical object and the network of relations supported between those objects, a *functor* is a principled way of comparing categories, translating the objects and actions of one category into objects and actions in another category in such a way that certain structural relations are preserved. Roughly, you can think of a functor as doing (any of) the following: specifying data locally; producing a picture of the source category in the target category, modeling one category or some aspect of that category within another; "realizing" an abstract theory of some structured notion (such as a 'group') in a certain background or on a specific "stage"; taking advantage of the methods available in the target category to analyze the source category; converting a problem in one category into another where the solution might be more readily apparent; forgetting or deliberately losing some information, perhaps in order to examine or identify those features more robust to variations or to ease computation. Formally,

Definition 6.0.2. A *(covariant) functor* $F : \mathbf{C} \to \mathbf{D}$ between categories $\mathbf{C}$ and $\mathbf{D}$ assigns to each object $c \in \mathbf{C}$ an object $Fc \in \mathbf{D}$ and to each morphism $f : c \to c' \in \mathbf{C}$ a morphism $Ff : Fc \to Fc' \in \mathbf{D}$. Moreover, these assignments must satisfy the following two axioms:

1. For any composable pair f, g in $\mathbf{C}$, $F(g) \circ F(f) = F(g \circ f)$.
2. For any object c in $\mathbf{C}$, $F(id_c) = id_{Fc}$.

A *(contravariant) functor*, i.e., a functor $F : \mathbf{C}^{op} \to \mathbf{D}$, is defined in the same way on objects, but differently on morphisms (reversing the direction of all arrows). All the information of this definition is displayed below (the covariant case on the left and contravariant case on the right, and with identity maps omitted, for readability, except for one object):

$$
\begin{tikzcd}
c \arrow[loop left, "\mathrm{id}_c"] \arrow[d, "f"'] \arrow[dd, bend left, "g\circ f"] & F(c) \arrow[loop left, "F(id_c)"] \arrow[d, "F(f)"'] \arrow[dd, bend left, "F(g\circ f)"] \\
c' \arrow[d, "g"'] & F(c') \arrow[d, "F(g)"'] \\
c'' & F(c'') \\
\mathbf{C} \arrow[r, "F"] & \mathbf{D}
\end{tikzcd}
\qquad
\begin{tikzcd}
c \arrow[loop left, "\mathrm{id}_c"] \arrow[d, "f"'] \arrow[dd, bend left, "g\circ f"] & F(c) \arrow[loop left, "F(id_c)"] \\
c' \arrow[d, "g"'] & F(c') \arrow[u, "F(f)"] \\
c'' & F(c'') \arrow[u, "F(g)"] \arrow[uu, bend right, "F(g\circ f)"'] \\
\mathbf{C}^{op} \arrow[r, "F"] & \mathbf{D}
\end{tikzcd}
$$

The truth of the frequently-cited claim of Eilenberg and Maclane that "the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation"[3] proves itself with time to anyone who works with categories. In addition to their intrinsic interest, functors are of special interest to us because of their essential role in the definition of *presheaves.*

Definition 6.0.3. A (set-valued) *presheaf* on $\mathbf{C}$, for $\mathbf{C}$ a small category, is a (contravariant) functor $\mathbf{C}^{op} \to \mathbf{Set}$.[4]

While a presheaf accordingly can initially be thought of as consisting of some specification or assignment of local data, according to the "shape" of the domain category, a sheaf will emerge as a special sort of presheaf in that its local data can be glued or patched

[3]See MacLane and Eilenberg, "General Theory of Natural Equivalences," 247.

[4]By "small," one means that the category has no more than a set's worth of arrows. Incidentally, the reader who wonders why, if a presheaf is just a (contravariant) functor, we bother giving it two names, might be satisfied by the notion used by the nLab authors of a *concept with an attitude.* This is meant to capture those situations when one and the same concept is given two different names, one of the names indicating a specific perspective or "attitude" suggesting what to do with the named thing. In renaming a particular contravariant functor as a presheaf, then, we have a concept with an attitude, specifically looking forward to *sheaves.*

together (locally). Before addressing in more detail the nature of presheaves via four principal perspectives on the action of the *presheaf-as-functor*, and then turning to sheaves, a few examples of functors are provided and the key notion of natural transformations introduced.

Example 6.0.3. In many settings, one might want to transfer one system of objects that present themselves in a certain way in one context to another context where irrelevant or undesirable (e.g., noisy) features are suppressed, while simultaneously preserving certain basic qualitative features. In the definition of a category, and indeed in the definition of many mathematical objects, typically one specifies (i) underlying data, together with (ii) some extra structure, which in turn may satisfy (iii) some properties. One obvious thing to do when considering some category $\mathbf{C}$ is to deliberately "forget" or ignore some or all of the structure or the properties carried by the source category by passing, via a functor, to another category. This process informally describes what are usually called *forgetful functors*, which provide us with a large source of examples.

There are many examples of this where **Set** is the target category, since many important categories are sets with some structure (however, forgetful functors need not have **Set** for the target category). While the "forgetting" terminology might vaguely suggest some sort of (possibly pejorative) loss of information, another way of looking at the same process is as extracting and emphasizing only the "important" features of the objects under study. An illustration of this comes from *detectors*, which do indeed act to forget or lose information, while preserving fundamental features of the underlying signal, and this is regarded as exactly what is useful about such tools, since what is removed is "clutter," leaving us with a compressed representation of the original information (with the effect that the result of applying the functor might be more robust to variations, more relevant to a particular application, simpler for computation purposes, etc.).[5]

We know that a *signal* is a collection of (local) measurements related to one another, and the topology on these measurements tells us how a measurement responds to noise, e.g., a signal over a discrete set is typically either not changed by noise at all or it changes drastically, while a signal over a smoother space may depend less drastically on perturbations. As already anticipated, as a forgetful functor, a detector acts to *remove* something—specifically, it acts to remove the topological structure from the signal. For a specific instance, consider a *threshold detector*. A threshold detector takes a continuous function $f \in C(\mathbb{R})$ and returns the open set on which $f(x) > T$ for some threshold $T \in \mathbb{R}$. The domain of this functor is the category with continuous functions for objects and for morphisms the functions $f \to g$ whenever $f(x) > g(x)$ for all $x \in \mathbb{R}$. The threshold detector is then the functor F that assigns to each $f \in C(\mathbb{R})$ the open set $F(f) = \{x \in \mathbb{R} : f(x) > T\}$, i.e., it lands in the sub-category (of "open sets") **Op** of **Set**. Moreover, one can see that if $f \to g$, then we will have $F(g) \subset F(f)$, making F a contravariant functor.

Example 6.0.4. Let $\mathbf{C}$ be an arbitrary category, and fix an object a of $\mathbf{C}$. Then we can form the (covariant) *Hom-functor* $Hom_{\mathbf{C}}(a, -) : \mathbf{C} \to \mathbf{Set}$, which takes each object b of $\mathbf{C}$ to the set $Hom_{\mathbf{C}}(a, b)$ of $\mathbf{C}$-arrows from a to b, and takes each $\mathbf{C}$-arrow $f : b \to c$ to the following map between hom-sets:

$$Hom_{\mathbf{C}}(a, f) : Hom_{\mathbf{C}}(a, b) \to Hom_{\mathbf{C}}(a, c), \tag{6.1}$$

[5] This example is derived from Robinson, *Topological Signal Processing*, 88-89.

which outputs $f \circ g : a \to c$ for input $g : a \to b$. In other words, the action on morphisms is given by *post-composition*.[6] Intuitively, the set $Hom_{\mathbf{C}}(a, b)$ can be thought of as the set of ways to pass from a to b within $\mathbf{C}$, or the set of ways a "sees" b within the context or framework of $\mathbf{C}$. Then, refraining from "filling in" the object b, it should be obvious how $Hom(a, -)$ can be thought of as representing in a rather general fashion 'where and how a goes elsewhere' or 'how a sees its world'. Given an object $a \in \mathbf{C}$, we say that the covariant functor $Hom(a, -)$ is *represented by* a; for reasons we will see below, this functor is also denoted Y^a. Moreover, it will turn out to be a massively important observation that instead of restricting ourselves to the hom-functor on some *given* a, we can assign to *each* object $c \in \mathbf{C}$ its hom-functor $Hom(c, -)$, and then collect all these together. Note that we can also form the *contravariant Hom-functor* $Hom_{\mathbf{C}}(-, a) : \mathbf{C}^{op} \to \mathbf{Set}$, for a fixed object a of $\mathbf{C}$, which works as one might expect. This functor can be thought of as representing 'how a is seen by its world'. Given an object $a \in \mathbf{C}$, we say that the contravariant functor $Y_a := Hom(-, a)$ is *represented by* a.

Example 6.0.5. Categories have underlying graphs. There is the important notion of a *diagram* in a category $\mathbf{C}$, a notion that in some sense captures a generalized idea of a subgraph of a given category's underlying graph. A *diagram* is defined as a functor $F : \mathbf{J} \to \mathbf{C}$ where the domain category $\mathbf{J}$, called the *indexing category* or *template*, is a small category. Typically, one thinks of the indexing category as a directed graph, i.e., some collection of nodes and edges, serving as a template defining the shape of any realization of that template

[6]This hom-functor will be defined for any object whenever the hom-sets of $\mathbf{C}$ are *small*. The other way of saying this is 'whenever $\mathbf{C}$ is *locally small*.'

in **C** and which may also specify some commutativity conditions on the edges which are to be respected by **C**. Then a diagram can be regarded as something like an instantiation or realization of a particular template **J** in **C**. Each node in the underlying graph of the indexing category is instantiated with the objects of **C**, while each edge is instantiated with a morphism of **C**. Functoriality demands that any of the composition relations (in particular, commutative diagrams) that obtain in **J** carry over (under action of F) to the image in **C**.

For some concrete illustrations of this, consider for our indexing category **2** (isomorphic to the linear order [**1**]), i.e., the category

$$\boxed{\begin{tikzcd} \bullet 0 \arrow[loop above, "id_0"] \arrow[r, "f"] & \bullet 1 \arrow[loop above, "id_1"] \end{tikzcd}}.$$

With such a category for indexing category, a (set-valued) diagram will yield a category that has as objects all the functions from one set to another set, and as morphisms the commutative squares between those arrow-objects. If we were to instead take as indexing category **3**, or [**2**], the linear order category with length 2,

$$\boxed{\begin{tikzcd} \bullet 0 \arrow[r, "f"] \arrow[dr, "g \circ f"'] & \bullet 1 \arrow[d, "g"] \\ & \bullet 2 \end{tikzcd}}$$

this would just act to pick out as objects commutative triangles. Finally, taking the category $\mathbf{2} \times \mathbf{2} \times \mathbf{2}$ as our indexing category just serves to pick out as objects commutative cubes in the target category:

$$\begin{tikzcd}
\bullet \arrow[rr] \arrow[dr] \arrow[dd] & & \bullet \arrow[dr] \arrow[dd] & \\
& \bullet \arrow[rr] \arrow[dd] & & \bullet \arrow[dd] \\
\bullet \arrow[rr] \arrow[dr] & & \bullet \arrow[dr] & \\
& \bullet \arrow[rr] & & \bullet
\end{tikzcd}$$

Functors are important for many reasons. As we shall briefly discuss below, universal properties are given in terms of functors. However, perhaps most important for our present purposes is the fact that *functors can be composed*; thereby, together with the trivial identity functor, we can produce *functor categories.*

There may exist a variety of ways of embedding or modeling or instantiating one category within another, i.e., there may exist many functors from one category to another. Sometimes these will be equivalent, but sometimes not. Moreover, the same blueprint may be realized in different ways, i.e., there can be different functors that act the same way on objects. *Natural transformations* enable us to compare these realizations. If functors allow us to systematically import or transform objects from one category into another and thus translate between different categories, natural transformations allow us to compare these different translations in a controlled manner.[7]

Definition 6.0.4. Given categories $\mathbf{C}$ and $\mathbf{D}$ and functors $F, G : \mathbf{C} \to \mathbf{D}$, a *natural transformation* $\alpha : F \Rightarrow G$, depicted in terms of its boundary data by the globular diagram

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left=50, "F"] \arrow[r, bend right=50, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathbf{D} \end{tikzcd}$$

consists of the following:

[7] The reader for whom categories are relatively new or exotic should not worry too much, on a first read through, about the following definition (and is free to skip it, resuming with the paragraph "Via..." on the next page); the important "take away" is just that functors can be treated as objects of a category, which are then compared via natural transformations.

- for each object $c \in \mathbf{C}$, an arrow $\alpha_c : F(c) \to G(c)$ in $\mathbf{D}$, called the c-component of α, the collection of which (for all objects in $\mathbf{C}$) define the *components* of the natural transformation;

- and for each morphism $f : c \to c'$ in $\mathbf{C}$, the following square of morphisms, called the naturality square for f, must commute in $\mathbf{D}$:

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"'] & G(c') \end{tikzcd}$$

The set of natural transformations $F \to G$ is sometimes denoted $Nat(F, G)$.

Composition of natural transformations is a little more complicated than the "usual" composition, for there are in fact two types of composition: vertical and horizontal:

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left=60, "F"] \arrow[r, "G"] \arrow[r, bend right=60, "H"'] & \mathbf{D} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "G_1"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{D} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbf{E} \end{tikzcd}$$

Vertical composition uses the symbol '$\circ$', giving $\beta \circ \alpha : F \Rightarrow H$ for the diagram on the left. Componentwise, this is defined by $(\beta \circ \alpha)_c := \beta_c \circ \alpha_c$. Horizontal composition uses the symbol '$\diamond$', giving $\beta \diamond \alpha : F_2 \circ F_1 \Rightarrow G_2 \circ G_1$ on the right, whose component at $c \in \mathbf{C}$ is defined as the composite of the following commutative square:

$$\begin{tikzcd} F_2F_1(c) \arrow[r, "\beta_{F_1c}"] \arrow[d, "F_2(\alpha_c)"'] \arrow[dr, dashed, "(\beta\diamond\alpha)_c"] & G_2F_1(c) \arrow[d, "G_2(\alpha_c)"] \\ F_2G_1(c) \arrow[r, "\beta_{G_1c}"'] & G_2G_1(c) \end{tikzcd}$$

For more details on the definition or facts about natural transformations, we refer the reader to any introductory text on category theory.

Via the notion of natural transformation, we can form the *functor category*, for natural transformations can be thought of as morphisms between functors. This is defined on categories $\mathbf{C}$ and $\mathbf{D}$ as having for objects all the functors from $\mathbf{C}$ to $\mathbf{D}$ and for morphisms all the natural transformations between such functors. There are clearly identity natural transformations and a well-defined composition formula for the natural transformations, and the category laws hold more generally, so we indeed have defined a category: the category of functors, denoted $Fun(\mathbf{C}, \mathbf{D})$, or more commonly, $\mathbf{D}^{\mathbf{C}}$. For our purposes, the most important thing to note here is that since presheaves are just another name for (contravariant) functors, and a morphism of presheaves from F and G just a natural transformation $\alpha : F \Rightarrow G$, we can form the presheaf functor category.

Definition 6.0.5. The *presheaf category*, denoted $\mathbf{Set}^{\mathbf{C}^{op}}$, is the (contravariant) functor category having for objects all functors $F : \mathbf{C}^{op} \to \mathbf{Set}$, and for morphisms $F \to G$ all natural transformations $\theta : F \Rightarrow G$ between such functors. Such a θ assigns to each object c of $\mathbf{C}$ a function $\theta_c : F(c) \to G(c)$, and does so in such as way as to make all diagrams

$$\begin{tikzcd} F(c) \arrow[r, "\theta_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(d) \arrow[r, "\theta_d"'] & G(d) \end{tikzcd}$$

commute for $f : d \to c$ in $\mathbf{C}$.

Natural transformations also allow us to define the important notion of *representable functors*. For a locally small category $\mathbf{C}$, we say that a functor $F : \mathbf{C} \to \mathbf{Set}$ is a *representable functor* if there exists an object $c \in \mathbf{C}$ together with a natural isomorphism $Hom_{\mathbf{C}}(c, -) \cong F$. If F is a contravariant functor, then the desired natural isomorphism is given by $Hom_{\mathbf{C}}$(-$-, c) \cong F$. We often write $Y^c = Hom_{\mathbf{C}}(c, -)$ in the covariant case, and $Y_c = Hom_{\mathbf{C}}(-, c)$ in the contravariant case, a notation that will be explained below.

In the covariant case, intuitively, the representable functor can be thought of as encoding how a category "is seen" or "is acted on" by a certain object; in the contravariant case, how the category "sees" or "acts on" the chosen object. For instance, in the category of topological spaces **Top**, if we regard all the maps from **1** (the one-point space) to a space X, this just produces the points of X, i.e., "**1** sees points".[8] The great importance of representable functors is in part due to the fact that representable functors can encode a universal property of its representing object. For instance, a category $\mathbf{C}$ will have an *initial object* if and only if the constant functor $* : \mathbf{C} \to \mathbf{Set}$ is representable, i.e., an object $c \in \mathbf{C}$ will be initial if and only if the functor Y^c is naturally isomorphic to the constant functor sending every object to the singleton set. Dually, an object $c \in \mathbf{C}$ will be *terminal* iff the functor Y_c is naturally isomorphic to the constant functor $* : \mathbf{C}^{op} \to \mathbf{Set}$. Put otherwise: an object $c \in \mathbf{C}$ is *initial* if, for all objects $d \in \mathbf{C}$, there exists a unique morphism $c \to d$; while an object $c \in \mathbf{C}$ is *terminal* if, for all objects $d \in \mathbf{C}$, there exists a unique morphism $d \to c$. It also turns out that all universal properties can be captured by the fact that certain data defines an initial or terminal object in an appropriate category, specifically the category of elements of the representable functor.[9] All of this is closely connected to perhaps the most important result in category theory, which makes good use of the above notion of natural transformations:

Proposition 6.0.1. (*Yoneda Lemma*) For any functor $F : \mathbf{C} \to \mathbf{Set}$, where $\mathbf{C}$ is a locally small category, and for any object $c \in \mathbf{C}$, the natural transformations $Y^c \Rightarrow F$ are in

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[8]This example is lifted from Leinster, *Basic Category Theory*.

[9]See my *Sheaf Theory through Examples* or any text on category theory for this construction.

bijection with elements of the set $F(c)$, i.e.,

$$\mathrm{Nat}(Y^c, F) \cong F(c). \tag{6.2}$$

Moreover, this correspondence is natural in both F and c. In the contravariant case, i.e., for $F : \mathbf{C}^{op} \to \mathbf{Set}$, things are as above, except we have

$$\mathrm{Nat}(Y_c, F) \cong F(c). \tag{6.3}$$

The idea here is that for a fixed locally small category $\mathbf{C}$, given an object $c \in \mathbf{C}$ and a (contravariant) functor $F : \mathbf{C}^{op} \to \mathbf{Set}$, then we know that the object c gives rise to another functor $Y_c : \mathbf{C}^{op} \to \mathbf{Set}$. A very natural question to ask, then, is about the maps $Y_c \Rightarrow F$,

$$\begin{tikzcd} \mathbf{C}^{op} \arrow[r, bend left=50, "Y_c"{name=U}] \arrow[r, bend right=50, "F"'{name=D}] & \mathbf{Set} \arrow[from=U, to=D, phantom, "\Downarrow ?"] \end{tikzcd}$$

The set of these natural transformations is just $Hom_{\mathbf{Set}^{\mathbf{C}^{op}}}(Y_c, F)$. But what is this set? Notice that from the input data F and c we were given, we could have also constructed the set $F(c)$. The Yoneda lemma just assures us that these two sets are the same; moreover, all the generality of natural transformations is encoded in the particular case of identity maps (used in the proof of the lemma).

Perhaps the most significant application of the Yoneda lemma is given by the *Yoneda embedding*, which serves to completely characterize natural transformations between representable functors, i.e., any (locally small) $\mathbf{C}$ will be isomorphic to the full subcategory of

$\mathbf{Set}^{\mathbf{C}^{op}}$ spanned by the contravariant representable functors, while $\mathbf{C}^{op}$ will be isomorphic to the full subcategory of $\mathbf{Set}^{\mathbf{C}}$ spanned by the covariant representable functors. We have seen that for each $c \in \mathbf{C}$, we have the covariant functor Y^c going from $\mathbf{C}$ to $\mathbf{Set}$ and the contravariant functor Y_c going from $\mathbf{C}^{op}$ to $\mathbf{Set}$. If we let this functor vary over all the objects of $\mathbf{C}$, the resulting functors can be gathered together into the (for example, covariant) functor $Y^{\bullet} : \mathbf{C}^{op} \to Hom(\mathbf{C}, \mathbf{Set})$. We also have the contravariant functor Y_c going from $\mathbf{C}^{op}$ to $\mathbf{Set}$, and collecting these functors together as we let c vary will give a functor $Y_{\bullet} : \mathbf{C} \to Hom(\mathbf{C}^{op}, \mathbf{Set})$.[10]

Definition 6.0.6. The *Yoneda embedding* of $\mathbf{C}$, a locally small category, supplies functors

$$
\begin{tikzcd}
\mathbf{C} \arrow[r, hook, "\mathbf{y}"] & \mathbf{Set}^{\mathbf{C}^{op}} \\
c \arrow[r, mapsto] \arrow[d, "f"'] & Hom(-,c) \arrow[d] \\
d \arrow[r, mapsto] & Hom(-,d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{C}^{op} \arrow[r, hook, "\mathbf{y}"] & \mathbf{Set}^{\mathbf{C}} \\
c \arrow[r, mapsto] \arrow[d, "f"'] & Hom(c,-) \\
d \arrow[r, mapsto] & Hom(d,-) \arrow[u]
\end{tikzcd}
$$

defining full and faithful embeddings.[11]

One can think of the Yoneda embedding $\mathbf{y}$ as a representation of $\mathbf{C}$ in a category of set-valued functors and natural transformations on some index category. An important consequence of the embedding is that any pair of isomorphic objects $a \cong b$ in $\mathbf{C}$ are representably isomorphic, i.e., $Y^a \cong Y^b$. The Yoneda lemma supplies the converse, namely if either the (co- or contravariant) functors represented by a and b are naturally isomorphic, then a and b will be isomorphic; so in particular, if a and b represent the same functor, then

[10] It is not unusual to rename these functors, as we do in the following definition, with a lowercase (bold) $\mathbf{y}$ in both cases, leaving the appropriate variance to context.

[11] See Riehl, *Category Theory in Context* for a proof and the definition of full and faithful embedding.

$a \cong b$. In many cases, it will be easier or more revealing to give such an arrow $Y^a \to Y^b$ or $Y_a \to Y_b$ than to supply $a \to b$, for the category $\mathbf{Set}^{\mathbf{C}^{op}}$ in general has more structure than does $\mathbf{C}$. Thus we can use the more advanced tools and universal properties that come with the presheaf category, and be sure that an arrow of the form $Y_a \to Y_b$, for instance, comes from a unique $a \to b$ even if $\mathbf{C}$ on its own may not allow the advanced constructions. Passing from a category $\mathbf{C}$ to its presheaf category can also be regarded as adjoining colimits (to be discussed in a moment, but think generalized sums) to $\mathbf{C}$, and doing so in the most "free" way, since in passing to the presheaf category, many non-representable ("ideal") presheaves will show up as well. As Awodey notes (echoing Hilbert's 1927 address "On the Infinite"[12]):

> [T]he category $\mathbf{Sets}^{\mathbf{C}^{op}}$ is like an extension of $\mathbf{C}$ by "ideal elements" that permits calculations which cannot be done in $\mathbf{C}$. This is something like passing to the complex numbers to solve equations in the reals, or adding higher types to an elementary logical theory. (Awodey, *Category Theory*, 167)

Philosophical Pass (1st Pass): Yoneda and Relationality

The Yoneda lemma tells us roughly that if there is a natural way of passing an object c's vision of its world (or how it is seen by its world) on to a functor F on that same category, then to recover this vision it suffices to ask F how it acts on c. While, mathematically speaking, the usefulness of the lemma often boils down to the fact that we are able to reduce the computation of natural transformations (which can be unwieldy) to the simple evaluation of a (set-valued) functor on an object, in a sense the full philosophical significance of the lemma points in the other direction. Given a category and an object in that category, rather

[12]This can be found in Benacerraf and Putnam, *Philosophy of Mathematics*.

than regard the object "on its own" (moreover, treating the entire category in a "detached" manner, as delimiting the outer boundaries of our consideration, or as "just sitting there"), via Yoneda we can regard that object as entirely characterized by its perspective or action on its world (or its world's perspective or action on it), and moreover place the category in which it lives in the wider category of all presheaves or sets varying over that category. The representable functors Y_c (or Y^c) present c in terms of all its interrelations with its context or world. The representable functor for an object c just captures, all at once, the most generic and universal "picture" of that object, supplying a "placeholder" for each of the possible attributes of that object (and then Yoneda's lemma just says that to specify an actual object of type c, it suffices to fill in all the placeholders for every attribute found in the generic thing of type c).

In this approach, what an object *is* can be entirely encapsulated by regarding "all at once" (generically) all of its interrelations with the other objects of its world. In the covariant case, we do this by regarding its actions on (or ways of affecting) other things; in the contravariant case, by its particular ways of being acted on (or being affected) by the other objects that inhabit its world. One might relate this to Spinoza's idea that what a body *is* is inseparable, in an important sense, from that body's particular *power to act and be acted on by other bodies*. We could also describe the idea more phenomenologically by considering, for instance, how the trained musician does not try to "understand" a piece of music by considering each component note or pitch all on its own, and then somehow stringing these detached perceptions together; rather, when the musician hears or plays a note, they are already considering or "hearing" that note in all its relations with the other notes of a given piece or composition tradition or tuning system. For instance, a musician's ear might come to understand a note's role in a given piece by "hearing" the set of all the notes it precedes (covariant) or all the notes it is preceded by (contravariant). In this way, by regarding all the ways a given note relates to the other notes of its "world," the musician constructs a vision of the note that is no longer "detached," but is entirely determined by all its relations to the other objects of its world, thereby restoring a more "continuous" (in the sense of non-detached and relational or connected) perspective. It seems this is already a far more natural perspective on things, one that could even explain why many experienced musicians come to hear certain pitches, even in their isolation (e.g., a single note struck on the piano or pitch coming from a train, outside of any context of a musical work), as "wanting to go" somewhere in particular—one might almost say, the trained ear does not hear middle C as if in a vacuum but rather hears the pitch *generically* by already hearing Y_C (or Y^C).[13]

Somewhat more generally, the intuition behind this "mentality" is that to know or access an object it suffices to know or access how it can be transformed into different objects (whether they are of the same type or not), or how other objects (of the same type or not) transform into it. In this connection, another way to approach the fundamental idea is to consider how we ordinarily (in our everyday lives) come to know an object, e.g., your favorite coffee cup. Throughout our lives, we amass collections of observations of certain attributes over certain regions of space-time—say, the color of the cup in different lightings, its slight smell, its feel. But we never "have" the coffee cup as a whole! The Yoneda lemma basically acknowledges this fact and simultaneously resolves it in a clever way: Yoneda simply tells

[13]The composers Schoenberg and Schenker, in their pedagogical works, went so far as to speak of a tone's intense "biological urges" and how each tone is already seeking "to propagate itself" in particular directions.

us that what the coffee cup *is*, is nothing "in itself" but rather is to be found in all possible relations other entities "of its world" might have with it (or actions other entities might take on it), or it with them (or actions it might take on other entities).

Via Yoneda, we can perform this sort of passage from the detached consideration of an object (our futile attempts to "grasp" an object "in itself") to the consideration of all its interrelations with the other objects of its world for *every object* of a given category. In passing to this next level, we can think of ourselves as taking an entire category $\mathbf{C}$ that previously was itself being regarded in a "detached" manner (and thus *discretely*), and placing it in the more continuous context of the category of all the presheaves over $\mathbf{C}$. The category of presheaves over $\mathbf{C}$ into which $\mathbf{C}$ is embedded not only has certain desirable properties that the original category may lack, like possessing all categorical limits, but it can be understood (in both intuitive and in various technical ways) as providing the continuous counterpart to the "detached" consideration of the original category.

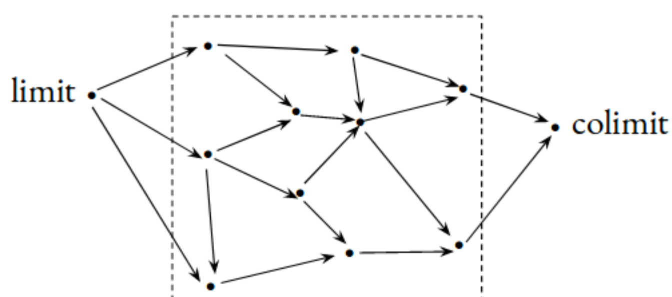
A Little More on Universality

There is a very important notion in category theory of limits and colimits. These really codify and powerfully generalize certain decisive constructions that had been noticed in many concrete cases all over mathematics well before category theory was born. While there is a fairly straightforward path to defining these precisely and in full generality (via cones and cocones), building on notions we have already seen, for the present audience the full formal details are perhaps unnecessary.[14] However, the main *idea* behind these notions—and some

[14]Either you do not already know about (co)limits, in which case, at this point, going down this particular rabbit hole risks exhausting you, or having you lose the bigger picture about sheaves; or you already know

of the ways they relate to (pre)sheaves and to a particular notion of *continuity*—remains a very important one, so the basic intuition will be discussed.

Basically, the notion of limit and colimit are defined in terms of arrows, so the fact that arrows have directionality accounts for why there are two sorts of limits (namely, a 'limit' and a 'colimit'). We saw earlier the notion of a *diagram* (as one example of a functor), namely an "instantiation" of a particular "template" supplied by the "shape" of the indexing category. You were invited to think about a template as consisting of some nodes and edges or arrows between certain of those nodes, i.e., as a directed graph. A diagram instantiates (in the target category) each node of the template with an object of the target category and each edge with an arrow of the target category, thus yielding a diagram built out of "shapes" provided by the template category. A limit and colimit are then given by nominated objects that "universally complete" the diagram both on the left and on the right (respectively), as in the following picture:



Each diagram in a given category can be thought of as posing two problems, the left and right problem, the solutions to which are supplied by certain objects (together with a collection of arrows) from which all arrows start (left solution) or at which all arrows terminate (right

something about (co)limits, in which case you know that there are plenty of places where one can find solid formal explanations of these notions.

solution), 'completing' the diagrams at either end.[15] In general, on any particular side, a solution need not exist at all (or it may exist on one side but not on the other); on the other hand, each problem may have many solutions. A *universal* solution is one through which each (left or right) solution must pass by means of a (fundamentally) unique mediating arrow. In other words, if there are solutions (of the relevant handedness), then the universal solution is one that is 'nearest' to the diagram, and as such, is the 'best' solution to the problem ('better' than any other object that can be used to complete the diagram). A limit is just a universal left solution, a colimit a universal right solution. If a diagram has a (co)limit, this (co)limit will be essentially unique, so whenever such a solution does exist, we can in fact speak of *the* (co)limit.

The notions of (co)limit are already of some "philosophical" significance, to be discussed in a moment in the next philosophical pass. But additionally, some of the important uses of such concepts lie in both a result concerning how we can think of presheaves in terms of colimits of a diagram of representable objects, and a further definition of (co)continuous functors, which we can take a moment to briefly discuss.

Proposition 6.0.2. Every object P in the presheaf category $\mathbf{Set}^{\mathbf{C}^{op}}$ is a colimit of a diagram of representable objects, in a canonical way, i.e.,

$$P \cong \mathrm{colim}\left(\int P \xrightarrow{\pi_P} \mathbf{C} \xrightarrow{\mathbf{y}} \mathbf{Set}^{\mathbf{C}^{op}}\right) \tag{6.4}$$

[15] Such "gadgets" formed by the object and arrows of the left solution—i.e., a special object c together with a collection of arrows, one for each object in the diagram, such that for any arrow between objects of the diagram, there are arrows from c that make the resulting triangles commute—is often called a *cone*. Similarly, for a right solution in which all arrows terminate, such a thing is often called a *cocone*.

where π is the projection functor from the category of elements of P and $\mathbf{y}$ is the Yoneda embedding.

Properly unpacking this would take us too far afield. However, roughly, this proposition states that given a (presheaf) functor $P : \mathbf{C}^{op} \to \mathbf{Set}$, there will be a canonical way of constructing a (small) indexing category $\mathbf{J}$ and a corresponding diagram $A : \mathbf{J} \to \mathbf{C}$ of shape $\mathbf{J}$ such that P is isomorphic to the colimit of A composed with the Yoneda embedding. That *every presheaf is a colimit of representable presheaves* is closely related to another construction, namely the *Cauchy completion* (or *Karoubi envelope*) of a category, in which the fact that representable presheaves are *continuous* in a precise sense is exploited. Without going into much detail, the main idea here is that while we have the powerful Yoneda (full and faithful) embedding that takes a category $\mathbf{C}$ to the category of presheaves $\mathbf{Set}^{\mathbf{C}^{op}}$, in general a category $\mathbf{C}$ cannot be recovered from $\mathbf{Set}^{\mathbf{C}^{op}}$; so a natural question to ask is how or to what extent, given $\mathbf{Set}^{\mathbf{C}^{op}}$, it can be said to determine $\mathbf{C}$. Basically, if a category $\mathbf{C}$ (or $\mathbf{C}^{op}$) can be shown to be *Cauchy complete*, then it not only can be recovered (up to equivalence) from the presheaf category (or covariant functor category of variable sets $\mathbf{Set}^{\mathbf{C}}$), but it can be shown to *generate* the original presheaf (variable set) category. We can recover a category given its category of presheaves by restricting precisely to the continuous presheaves there.[16] Recall that the Yoneda embedding of a category $\mathbf{C}$ into the category of presheaves $\mathbf{Set}^{\mathbf{C}^{op}}$ "freely" added all colimits to $\mathbf{C}$; one way to think about the Cauchy

[16]Where these are defined in terms of the *retracts* of representables.

completion $\overline{\mathbf{C}}$ of $\mathbf{C}$ is as lying *in between* this "free cocompletion" of $\mathbf{C}$, i.e.,

$$\mathbf{C} \hookrightarrow \overline{\mathbf{C}} \hookrightarrow \mathbf{Set}^{\mathbf{C}^{op}}.$$

For a small category $\mathbf{C}$, that the category is Cauchy complete can be shown to be equivalent to stipulating that the category admits all small absolute colimits, where an absolute colimit is a colimit preserved *by any functor* whatsoever. In this connection, this also gives us the opportunity to supply the important (and more standard) definition of the following:

Definition 6.0.7. A functor is *(co)continuous* if it preserves all small (co)limits,

where "preservation" is defined as follows: for any class of diagrams $K : \mathbf{J} \to \mathbf{C}$ valued in $\mathbf{C}$, a functor $F : \mathbf{C} \to \mathbf{D}$ is said to *preserve* limits if for any diagram K and limit cone over K, the image of this cone under the action of the functor defines a limit cone over the (composite) diagram $F \circ K : \mathbf{J} \to \mathbf{D}$. Importantly, covariant representable functors preserve all limits, taking limits in $\mathbf{C}$ to limits in $\mathbf{Set}$; contravariant representable functors preserve all limits in $\mathbf{C}^{op}$, taking colimits in $\mathbf{C}$ to limits in $\mathbf{Set}$.[17] Intuitively, this concept of a (co)continuous functor as a special sort of functor (structure-preserving action) that takes universal objects in the source category to universal objects in the target category can be thought of as follows: whatever else the functor does to objects as it sends objects from one category to another, a (co)continuous functor will send the object that acted as a privileged mediator or gateway ((co)limit) in relation to the rest of the objects in the source category

[17] Accordingly, the Yoneda embedding $\mathbf{y} : \mathbf{C} \hookrightarrow \mathbf{Set}^{\mathbf{C}^{op}}$ preserves all limits that exist in $\mathbf{C}$; and the embedding $\mathbf{y} : \mathbf{C}^{op} \hookrightarrow \mathbf{Set}^{\mathbf{C}}$ preserves limits in $\mathbf{C}^{op}$. It will turn out that a sheaf is a special sort of presheaf that preserves limits in just this way.

to an object that similarly plays the role of privileged mediator for its fellows in the target category. We could label this characterization of continuity via the preservation of the roles of privileged mediators as *morphological continuity*.

Philosophical Pass (2nd pass): Universality and Mediation

At least in terms of where we are headed, perhaps the most important take-away from the preceding few pages is the notion of a (co)continuous functor as one that preserves all small (co)limits. This "morphological" understanding of continuity as a preservation of limits (universal objects) is important for certain definitions of sheaves to follow; additionally, the ideas of the preceding section seem to be of some intrinsic "philosophical" interest.

We know that a universal property is expressed by a representable functor and that each presheaf is a colimit (universality!) of a diagram of representables. Broadly, universality can here be understood to mean that the object occupies a privileged position in relation to other objects of its "world," in that it serves as a gateway or bridge for all other relations: "you have to pass (factor) through me if you want to relate to anything else." If such universality is thus thought of in terms of an object's privileged role as mediator or "hub" for all other objects of the same type trying to relate or interact, functorial continuity can broadly be understood in terms of how, in passing from one network of objects and relations to another, there is a *preservation* of the roles of those designated objects occupying a privileged position as mediator within their respective networks, i.e., in passing from one "world" to another, the special mediator in one "world" gets mapped, or has a "direct line," to the special mediator in the other world. As we saw in Chapters 1 and 2, Aristotle's *Analytics* and *Physics* initiated a number of important but still rather obscure connections between universality, "the middle" (mediation), and continuity, certain of which connections arguably resurface in Hegel's conjecture that ultimately continuity and derivability are one and the same (something he seems to have held precisely on account of the importance he placed, throughout all aspects of his system, on *mediation*) and then, in a very different context, in Charles Peirce's notion of "thirdness." In category theory, the profound connection between universality and mediation (the "gateway property") is made more precise. It is curious that in making this connection more exact, continuity (of the "morphological" sort) also re-emerges in a pivotal way.

We also saw that, complementing the determination of continuity via the Yoneda embedding, (going in the opposite direction) we could recover a category from its category of presheaves by restricting our attention precisely to the so-called continuous objects there. In brief, then, connections between continuity, universality, and *mediation* were established along three main lines: (i) in the "gateway" understanding of (co)limits; (ii) via the fact that every presheaf is a colimit (universality) of a diagram of representables; and (ii) via the recovery of a category from the category of presheaves on that category through restriction to the continuous presheaves (retracts of representables), which lie "in between" the category itself and the category of presheaves on it.

Perhaps most important of all, we saw the pivotal notion of a *(co)continuous functor* as a special sort of functor (structure-preserving action) that takes universal objects in the source category to universal objects in the target category. In other words, continuity is envisioned as a special kind of passage or translation from one "world" of objects to another "world" of objects—*special* in that, in passing between worlds, it takes care to preserve the special role of those that act as privileged mediators for the rest of the objects of their world.

More on the Presheaf Category

Before introducing sheaves, let us consider more closely the *action* of a presheaf. For a presheaf $F : \mathbf{C}^{op} \to \mathbf{Set}$, it is often natural to think of the result of applying F as some sort of "container" of $\mathbf{C}$-shaped "figures," where the objects c of $\mathbf{C}$ are regarded as supplying the "generic figures" or "shapes" that get *instantiated* in $\mathbf{Set}$, i.e., $F(c)$ is some particular set of instantiations or figures of c-shape. Another first and common way of thinking about how the presheaf acts on objects, especially natural in certain contexts, is that for each object c of $\mathbf{C}$, the functor specifies a set $F(c)$ regarded as the set F *taken at stage* c.

Especially for philosophers, there is some utility, at least initially and heuristically, in thinking about the above in terms of Plato's concept of the *form* or *shape* (*eidos*) of something as what that thing "really is"—as at once invariant, sufficiently generic, and fundamentally "simpler" (and, as such, more *intelligible*)—but which also supports a great variety of realizations or instantiations in a plurality of particular and changeable "appearances" (*phantasmata*) of it (through a process Plato would call the "participation" (*methexis*) of the form in the appearances). To be perfectly explicit, the idea here is that the "generic figures" supplied by $\mathbf{C}$ act as something like the "form," while the value assignments $F(c)$ for each object of $\mathbf{C}$ supply something like the "appearances" or "instantiations" of the form "in time," and while the presheaf F itself is nothing other than this process of realization or participation-in-time of the form.

But so far, this is just to consider how the presheaf operates *on objects.* Obviously, as a functor, we must also consider the (right) action specified by the (contravariant) functor, i.e., how it acts *on morphisms.* The basic idea here, following the "figures" of "generic shape" c interpretation, is that a morphism in $\mathbf{C}$ from one object to another will give rise to a "change of figures," where this means, more precisely, that if we have a figure x of shape c (i.e., $x \in F(c)$) and a figure y of shape c' (i.e., $y \in F(c')$), then asking about the effect of changes of figures amounts, at the level of the presheaf, to asking to what extent the figures are incident or overlap (and what this overlap structure looks like). Notice also that, adopting the interpretation of $F(c)$ as a set existing *at stage* c, the morphisms of $\mathbf{C}$, upon being acted on by the presheaf, would amount to a "variation in the stage," so that, overall, the presheaf can be interpreted as supplying a picture of a set *varying through time*. These are just some of the natural ways of beginning to think about presheaves and their action.

We will think of there being *four characteristic kinds of cohesivity or variability* presented by presheaf categories in accordance with four main ways the right action can be found to operate. The right action can be thought of as operating in any of the following four main ways:

1. as *processual*, e.g., as passing from sets indexed by one stage to sets indexed by another, giving rise to the notion of a $\mathbf{C}$-variable set, modeling sets evolving through time.[18]

2. as *extracting boundaries*, e.g., graphs with source and target map, simplices picking out lower dimensional boundaries. For something like a topological space that consists of 'points' and 'edges' and 'triangles', etc., in changing figures, we pass

[18]Objects of $\mathbf{C}$ play the role of stages; for every c in $\mathbf{C}$, the set $F(c)$ is the set of elements of F at stage c, while the morphisms model transitions between stages. (A presheaf on $\mathbf{C}$ is just a set varying over the category $\mathbf{C}^{op}$.) We can then form the category $\mathbf{Set}^{\mathbf{C}}$ with objects $\mathbf{C}$-variable sets, i.e., functors, and arrows the natural transformations between them.

from higher-dimensional figures to lower, so that, for instance, the action works by extracting the end-points of an edge or extracting the edges of a triangle, etc.

3. as *consistency conditions on how different "probes" of a space relate*, where a presheaf X in general is regarded as something like a rule assigning to each object U of $\mathbf{C}$ (each "test space") the set $X(U)$ of admissible maps from U into the space X, which space is thus progressively "probed" by the constituent shapes of the domain category, thereby being "modeled by" such probes and their interactions.
4. as *restriction*, e.g., whenever some sort of topology is involved, where the data specified over a "larger" region is restricted down to the data specified over a region included in the former region.[19]

In the interests of space, I focus on the fourth action perspective, giving two examples (that will be important in the later section on sheaves).[20]

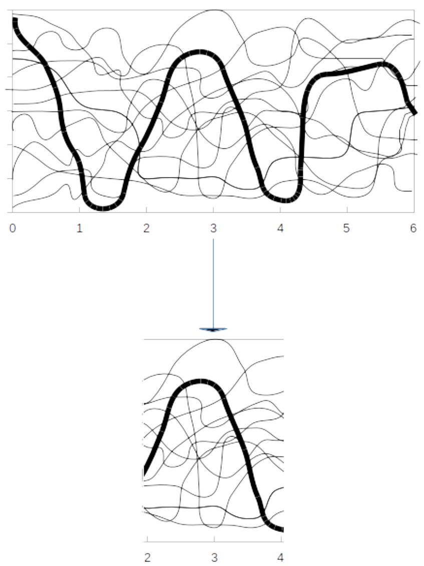
Example 6.0.6. To illustrate the last action perspective—namely, action by *restriction*—we can begin by constructing a presheaf on the lattice or partial order of open sets $\mathscr{O}(X)$, for X a topological space. A presheaf on X is just a functor $F : \mathscr{O}(X)^{op} \to \mathbf{Set}$. For each open $U \subseteq X$, we then think of the set $F(U)$ as the set that results from assigning values throughout or "over" U. An open $V \subseteq U$ is just an inclusion arrow in $\mathscr{O}(X)$, so applying the (contravariant) functor F gives us a function called the *restriction* from the values specified

[19] It is not uncommon to introduce and discuss presheaves and sheaves exclusively via this fourth approach, but the first three perspectives are also important to consider, especially since the first two often involve examples with finitely generated categories (and, as such, provide a good stock of simple and computationally tractable examples), and the third achieves a level of generality that ultimately lets us speak of sheaves in "higher dimensions."

[20] For the curious reader, my *Sheaf Theory through Examples* provides detailed discussions and a variety of examples for all four perspectives.

over U to the values over V, typically denoted $\rho_{V,U} : F(U) \to F(V)$. Given an $f \in F(U)$, we can denote $\rho_{V,U}(f)$ by $f|_V$, called the restriction of f from U to V.

As a first illustration of such a functor, we can consider the set of all continuous real-valued functions, i.e., functions from $U \subseteq X$ to $\mathbb{R}$. Importantly, when $V \subseteq U$, we have a restriction function $\mathbf{Top}(U, \mathbb{R}) \to \mathbf{Top}(V, \mathbb{R})$, which just sends $f : U \to \mathbb{R}$ to $f|_V : V \to \mathbb{R}$. We thus *restrict* our collection of functions given over some region (say, (0,6)) down to the open subsets of that region (say, (2,4)), as in the following general picture:



Thus the right action of this presheaf is given by *restriction*, and this action by restriction is clearly functorial.

Example 6.0.7. For another restriction-type example, but of a rather different flavor, take for regions the set of jurisdictions with their sub-jurisdictions, i.e., the set J so that $(J, \subseteq)$

forms a preorder.[21] We can consider that within the set of *possible laws*—where laws are just propositions, i.e., objects of the preorder **Prop** regarded as a category whose objects are logical propositions and whose morphisms are *proofs* that one statement implies another—some of these laws are being followed by all people in the region. To each jurisdiction V, then, we can assign a set $R(V)$ of the laws being respected by all the people throughout V. (In other words: laws are being assigned *locally* to each jurisdiction; after all, a law is dictated to be valid only within a specific region—and that was all we meant by calling something a "local" assignment of data.) If $V \subseteq U$, any law respected throughout U is obviously respected throughout V. In other words, we have that $R(V) \supseteq R(U)$ (note the reversal of direction). Clearly any law respected throughout all of Illinois will be respected throughout any county included in Illinois. But the converse is not true!

So we have a local assignment of data to the "space" of jurisdictions that moreover obeys the property that whenever we have an inclusion of a region V in a region U, then data assignments specified to hold throughout U are included in the set of data assignments holding throughout V—or, equivalently, whenever V is included in U, then we can *restrict* the data assignments specified to hold throughout V down to those given throughout U. The idea to keep in mind here is this: if you have some data (like a list of those laws being respected by everyone) assigned to some region (like Illinois), and you have another list of laws being respected by everyone in some subregion of that region (like Cook County), then you will expect that the list of laws respected by everyone throughout Cook County will be (equal if not) larger than the list of laws respected by everyone throughout Illinois. Think

[21] This example is derived from Spivak, *Category Theory for the Sciences*.

about it this way: in a larger region, there are more chances for the data "not to fit," e.g., for someone to fail to respect that law, than there are in a smaller region. The main take-away here is that we have made use of two key ingredients: (1) a local assignment of data to a space (the space of jurisdictions, ordered by inclusion); and (2) a natural operation of "restriction" (reversing the direction of inclusion) that takes the data attached to a region and restricts down to the data attached to a region that included the former region. Formally, these two ingredients just specify what we need to have a functor R that is *contravariant*, i.e., we have produced a functor $R : J^{op} \to \mathbf{Set}$.

Philosophical Pass (3rd): Four Actions

Many important mathematical structures and categories arise as a presheaf category consisting of functors on some given indexing category and landing in **Set**, where the result of applying the functor to the objects of the indexing category yields something we can naturally think of as a "container" (living in **Set**) holding various instantiations, each of which conform to the "shape" or "form" or "stage" determined by the "generic figures" populating the indexing category (one for each of its objects), and where the "changes of figure" indicated by the indexing category (given by its morphisms) act on, and are respected by, the figures instantiated in the container. This is a generally useful perspective on things, but is especially apt when the objects and morphisms of the indexing category $\mathbf{C}$ have some sort of geometrical interpretation.

The fundamental idea joining the four perspectives on the presheaf action described above is that the domain category $\mathbf{C}$ plays the role of specifying the general internal structure or nature of the test spaces, the figure-types or shapes, the glue, the temporal continuity (internal dynamic), or the locality of the data assignments in which all the sets in $\mathbf{Set}^{\mathbf{C}^{op}}$ must participate—and where the resulting total presheaf category has for its objects all the different instantiations or "realizations" that each exemplify or mobilize the overall form supplied by $\mathbf{C}$ in (possibly) different ways. The other way of looking at this is that the domain category plays the role of a parameter providing the *form* of how (temporal, dynamic, geometric) variation is to take place, while the target category (**Set**) serves as the container or arena holding on to all the particular values or results of "participations" in this form of variation.[22] One might think of a presheaf as mediating between the invariance or fixedness "outside of time" of the domain category and its multifarious presentations or appearances "in time." As such, a presheaf itself is not to be situated in the foreground or in the background, but should be understood as embracing the movement between the background and foreground.

It is worth emphasizing that while all the presheaf functors above are valued in **Set**, which is useful in 'taming' many problems, presheaves are anything but the static and uniform objects the usual naive set-theoretical perspective encourages. Considering presheaves with an action that is *processual* recaptures a dynamic perspective in which objects are not regarded as static collections but are seen as evolving through stages, either merely temporally (as in the case of registering the changes in the sets of notes being played on a keyboard from one moment to another) or in accordance with an internal dynamic (as in case of motions or evolutions of a certain shape, subject to certain equations, fixed

[22] Aspects of this might also remind the reader of Chapter 3's description of some of Oresme's innovations.

points, etc.). Against the generally "discrete" and static context of sets, this restores a more "continuous" perspective, in the form of dynamism or process.

The *boundary-extraction* perspective, for its part, allows us to see how in certain instances the boundaries (figures of a certain type) of figures of a certain type can be extracted, which in turn reveals the incidence relations that, altogether, describe something like how the structure "holds together." As against the nature of objects grouped together more or less arbitrarily into a set that cannot internally differentiate objects or discern important qualitative features of those objects or their modes of relation, this perspective restores a kind of continuity in the form of qualitative differentiation. The third perspective is far more involved, and full discussion of it is omitted, in the interest of space. But note that it lets us regard a space in terms of ways of "probing" it from the "outside" and thinking about the entire space in terms of how these various probes behave with respect to one another.

Finally, acting via *restriction*, presheaves open onto a range of *relationships between the parts of a whole*. In general, such relationships emerge as in some sense "regular," namely in that in passing from data over some containing region to data over a sub-region, there remains a kind of stability in the conformity of the contained parts to the same rule or function attached to the containing region. This perspective thus opens onto a form of conformity of parts to a single rule or idea (against the usual set-theoretical consideration of a "whole" independently of the specific way, beyond whether or not a part "belongs," that whole enforces certain relationships among the parts).

In short, via these general four presheaf action types, while we benefit from the "nice" and "tame" properties of **Set**, we simultaneously recapture a more "continuous" (in a variety of more or less intuitive senses) perspective in the form of (1) *dynamism* (replacing the static world of fixed objects in **Set**); (2) *extraction of what holds figures together* (reinstating the *qualitative* against an emphasis on quantitative differentiation); (3) presheaves on a category **C** of "spaces" as giving rise to spaces *modeled on* **C** in the sense that they are being "probed" by the objects of **C**; and (4) *regularity or conformity of parts to a single specific rule* (against the usual set-theoretical reduction of all unity to that of "belonging").

Section 3: Sheaves

Introducing Sheaves

The previous two examples of presheaves (continuous functions and jurisdictions) are in fact already *sheaves.* But in order to provide a first definition of a sheaf, we need to introduce one last notion, namely that of a *covering.* We will return to (and significantly generalize) this notion of covering in later sections; for now, we will just think of a cover in the context of topological spaces and open sets.

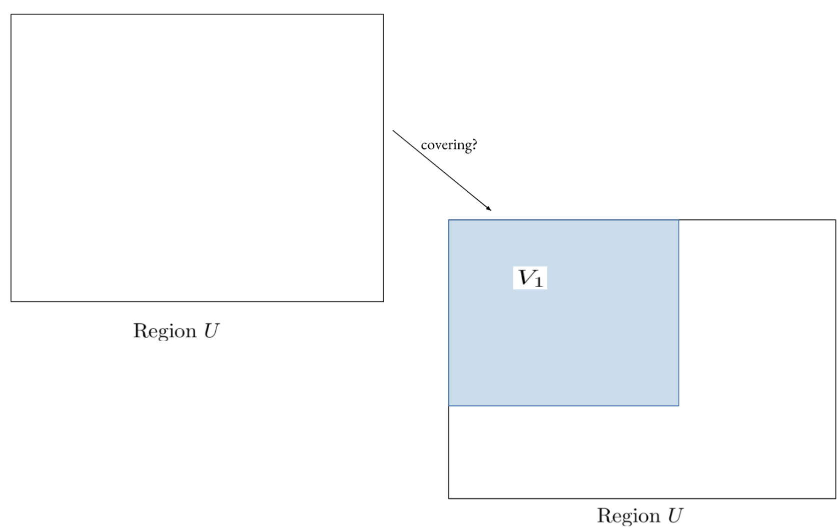
Definition 6.0.8. Given X a topological space and $U \subseteq X$ an open set of the space, consider $V_1, \ldots, V_n$ open subsets (think "subregions") of U, i.e., for all $1 \leq i \leq n$, $V_i \subseteq U$. Then the V_i are said to collectively *cover*, or provide a *covering* of, U if every point that is in U is in a V_i for some i.[23]

Roughly, but also looking ahead to the more general approach, one can think of a covering of a given object U in terms of a *decomposition* of that object into simpler ones, the resulting simpler "pieces" of which, when taken altogether, can be used to recompose all of U. In

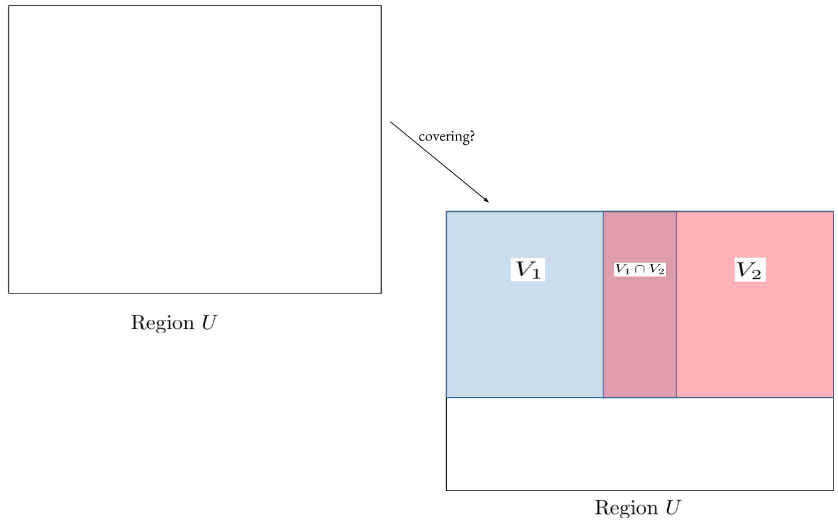
[23]Another way of saying this is that for $\mathscr{O}(X)$ the poset of open subsets of X, ordered by inclusion, an I-indexed family of open subsets $V_i \hookrightarrow U$ *covers* U provided the diagram consisting of the sets V_i together with the inclusions of all their pairwise intersections $V_i \cap V_j$ has U for its colimit.

terms of covers of a *set* U, this has a very simple description: it is just a family of subsets $\{V_i | i \in I\}$ for which their union is U itself, i.e., $\bigcup_{i \in I} V_i = U$. For now, intuitively, it is perfectly fine to just think of a covering in terms of specifying a collection of subregions that can be "laid over" a given region in such a way that the entire region is thereby covered. An obvious but decisive observation is that such subregions can overlap one another.

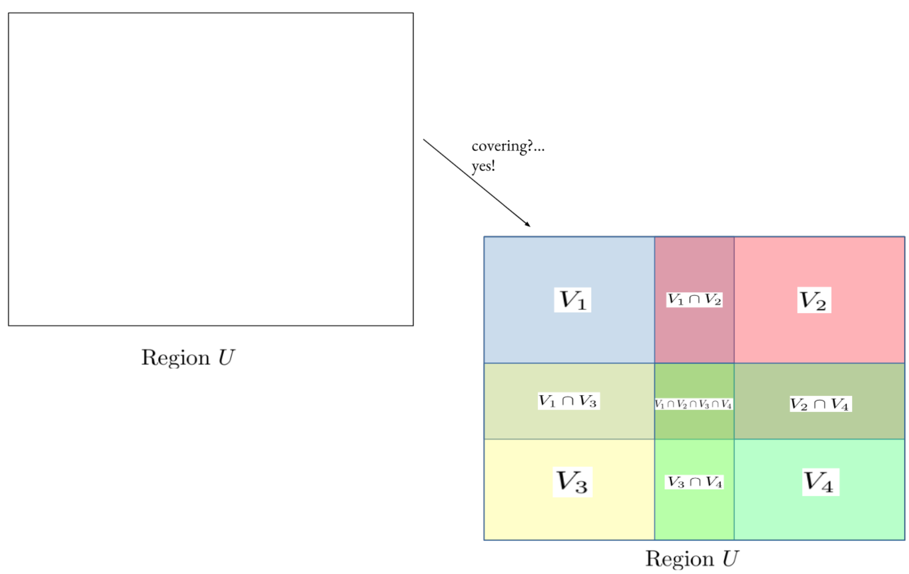
In case these notions are not already clear to the reader, the image to keep in mind is that we have a region U that we want to cover with some collection of "pieces" into which it has been "decomposed." We might try covering it with some $V_1 \subseteq U$, as follows:



V_1 alone clearly fails to cover U. We might try adding another region $V_2 \subseteq U$, thus:



Again, V_1 and V_2 together fail to cover U. Yet note that now there is a subregion where V_1 and V_2 overlap. We might continue in this manner, working our way up to a proper cover of all of U, such as that provided by the following collection V_1, V_2, V_3, and V_4:



The notion of a presheaf (or data being assigned locally to a "space"), together with the above notion of a covering of the space, enables us to offer a first pass at a definition of a sheaf, a definition very much motivated by the action-as-restriction presheaf perspective.

Definition 6.0.9. Assume given X a topological space, $\mathscr{O}(X)$ its partial order of open sets, and $F : \mathscr{O}(X)^{op} \to \mathbf{Set}$ a presheaf. Then given an open set $U \subseteq X$ and a cover $V_1, \ldots, V_n$ of U, for this cover we have the following *sheaf condition*:

- Given a sequence $a_1, \ldots, a_n$, where each $a_i \in F(V_i)$ is a value assignment given throughout V_i, whenever we have that for all i, j, $a_i|_{V_i \cap V_j} = a_j|_{V_i \cap V_j}$, then there exists a unique value assignment $y \in F(U)$ such that $y|_{V_i} = a_i$ for all i.

The presheaf F is then a *sheaf* whenever it satisfies this sheaf condition for every cover.[24]

Let us break this definition down into four, more easily "digestible," steps. The idea is this: given a presheaf on some space and a covering of that space, the definition of a sheaf begins by making use of what is sometimes called a *matching family*, defined thus:

Definition 6.0.10. A *matching family* $a_1, \ldots, a_n$ of sections over the $V_1, \ldots, V_n$ consists of a section a_i in $F(V_i)$—i.e., a value assignment throughout V_i, chosen from the entire set of value assignments given over V_i—for each i, such that for every i, j, we have

$$a_i|_{V_i \cap V_j} = a_j|_{V_i \cap V_j}.$$

In other words: given a data assignment a_i throughout region V_i and a data assignment a_j throughout region V_j, if there is agreement between the different data assignments over the sub-region where V_i and V_j overlap, then both data assignments a_i, a_j together give a

[24] An important terminological remark: elements of $F(U)$, i.e., value assignments specified over $U \subseteq X$, are called *local sections* of the sheaf F over U, while elements of $F(X)$, i.e., value assignments given over the *entire* space, are called *global sections* of F. More broadly, whenever local information, e.g., elements like functions f and g given over certain domains, restricts to the same element in the intersection of their domains, then such f and g are called *sections*.

matching family. As the definition requires that it holds for every i, j, the idea is that we can build up "large" matching families $a_1, a_2, a_3, \ldots, a_n$ from such pairwise checks for agreement.

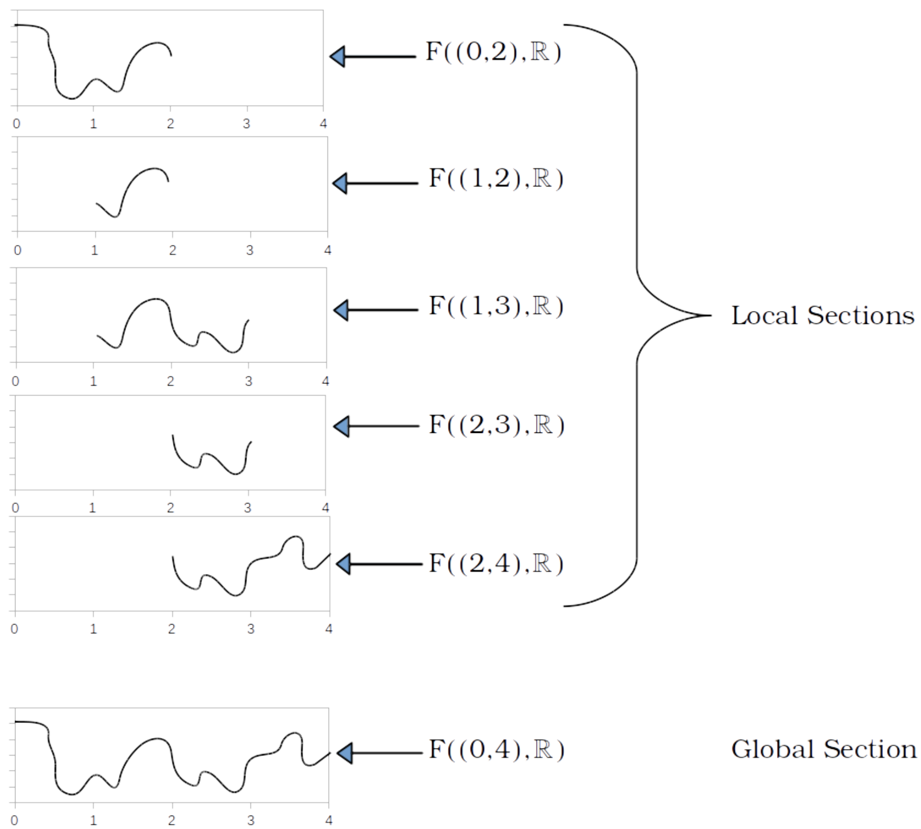
Next, the definition specifies what is sometimes called a *gluing*. Given a matching family for our cover of the space U, we call a value assignment throughout *all* of U a *gluing* if, whenever this (unique) data assignment given over the entire space U is restricted back down to the subregions that make up the cover, it is equal to the original local data assigned to each subregion.

With the notion of a matching family and that of a gluing, the sheaf definition is basically complete. The penultimate step is to spell out what it means for the sheaf condition given above—i.e., for a matching family, there exists a unique…, etc.—to be satisfied for every matching family. If, *for every matching family, there exists a unique gluing*, then we say that the presheaf F satisfies the *sheaf condition.* The final step involves stipulating that a presheaf is a sheaf whenever it satisfies this sheaf condition *for every covering.* That is all—we have defined what a sheaf is!

But the definition alone is probably not very illuminating to those who are seeing this for the first time. Thus, it is best to unpack the definition with examples. A variety of simple and guiding examples are given in what follows.

Initial Examples

Example 6.0.8. We return to the presheaf of continuous real-valued functions on a topological space X from the previous section. As mentioned, this is a sheaf, specifically a sheaf of real algebras associating to each open $U \subseteq X$ the algebra $F(U)$ of real-valued continuous functions defined there. Not only can we restrict functions to any open subset, but we can also glue together local assignments whenever they agree on overlapping regions, producing a global assignment, i.e., a consistent assignment *over the entire region* that will agree with the local assignments when restricted back down to each subregion. This process is nicely captured by an image of the following sort:



Example 6.0.9. Revisiting the example of the presheaf of laws being respected throughout a jurisdiction (a geographic area over which some legal authority extends): for X the entire world, to each jurisdiction $U \subseteq X$ we assigned the set $R(U)$ of laws being respected throughout the region U. Is this presheaf R a sheaf? Well, we can check: given some law respected throughout U and another law respected throughout V, do they agree on the sub-region where U and V overlap? In this case, we can basically just observe whether they amount to

the same law on the overlapping region $U \cap V$.[25] (If there is no overlap, then this is trivially satisfied.) Now repeat this check for each such pair of overlapping regions.

For instance, on U there might be a law that stipulates "no construction near sources of potable water," while on V a law might stipulate "no construction in public parks." If it turns out that on the overlapping subregion $U \cap V$ all public parks are near sources of potable water (and vice versa), then the laws agree on that overlapping region, and thus can be "glued" together to form a *single* law about construction that holds throughout the union $U \cup V$ of these two jurisdictions.

This might seem like a rather harmless or trivial construction, but consider that the global sections of such a sheaf R would tell you exactly those laws that are respected by everyone throughout the planet. This would be a useful piece of information! (It might reveal the sorts of "bedrock" shared values that are ultimately respected, in one form or another, by every society.) The process of "checking" for agreement on overlapping regions is straightforward, but the resulting observations or data assignments one can now make concerning the entire space, via the global sections, can be very powerful and far-reaching.

Example 6.0.10. The 20th-century pianist Glenn Gould was one of the first to ardently defend the merits of studio recording and use of the *tape splice* in the creative process, against those who held fast to the supposedly more "moral" or "pure" tradition of the live concert performance (and who accordingly thought that the only purpose of the splice would be to rectify performance mishaps or to alleviate the pressure of the "one-take" approach

[25] In fact, we do not really need that they are exactly the same law, just that there is a consistent system of "translation" between the sets of laws, i.e., a set of isomorphisms translating between each such pair.

demanded by the concert form).[26] Gould challenged the view that the only legitimate continuity of a unified interpretation could come from the one-takeness of traditional performance, proposing instead that the listener's "splice prerogative" and the performer's newfound editorial control in the recording studio would bestow upon creators an even more demanding ethic concerning matters of architecture and integrity of vision. Gould claimed that new, explicit, and more demanding forms of continuity were to be found in this montage-based approach: "splicing builds good lines, and it shouldn't much matter if one uses a splice every two seconds or none for an hour so long as the result *appears* to be a coherent whole." His wager was that just as one does not demand or expect that the filmmaker shoot a film in one shot, one should not expect that the coherence or continuity of an interpretation of a musical piece can only be secured by the inexorable linearity of time and the single take—the musician has just as much a right to montage as the filmmaker.

Gould went as far as to test, with a controlled experiment involving eighteen participants, whether listeners (including laymen and recording experts) could detect the "in point" of any splice in certain selections of recordings, each of which selection had drastically different splice densities (in some cases, none).[27] What he found, in short, was that "the tape does lie and nearly always gets away with it." While originally (with analog magnetic tape splicing) the tape splice involved careful (and literal) cutting of the physical tape with scissors or a blade and (literal) gluing or taping of it to another section of tape (possibly from an entirely different recording session)—whenever qualities, such as tempo, of the two

[26]By splices, one means an edit point representing the confluence of distinct takes or inserts (i.e., recorded performance of a portion of the score).

[27]The results can be found in his essay Gould, "The Grass is Always Greener in the Outtakes."

recordings to be joined could be made compatible enough to permit a seamless joining—Gould foresaw the power inherent in the more general notion of splicing and montage to at once provide a more analytically acute dissection of the minute connections ultimately defining the identity of a piece and a more explicit approach to the architectural continuity (one that left behind the "in-built continuity" allegedly belonging to the one-take concert ideal, and focused instead on breaking a piece down into its smallest parts, separately recording many "takes" of such portions, and then gluing together a certain selection of those diverse performances, whenever they could be made locally compatible along their overlap—all with the aim of producing a single, unified realization of the entire piece). Gould's insistence on the tape-splice and on the centrality of montage in recording practice nicely captures the fundamental "spirit" of the sheaf construction, and again suggests the close connection between sheaf theory and the transformation and refinement of the concept of continuity.[28]

Moreover, while the single-take approach to recording and unifying the musical idea is essentially *deductive* (and purports to be neutral in its simple "transmission of the facts"),

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[28]Incidentally, Gould's closeness to the sheaf "mentality" is evidenced in a number of aspects of his life, not just his art, for instance via his impressive insistence on forever integrating as many disparate planes and partial pieces of information as possible into a single coherent experience, e.g., his alleged habit of simultaneously listening to all of the conversations going on in a café. This mentality is perhaps most famously illustrated by the various accounts of him purposefully dividing his concentration across multiple channels in order to better understand something, as for instance when he claimed that he discovered he could best understand Schoenberg's Opus 23 if he listened to it while simultaneously playing the news on the radio, or when he mastered a demanding section of a Beethoven sonata only after placing a radio and television next to the piano and turning them up as loud as they would go as he worked through that passage. This embodies the sheaf philosophy: integration and coherence not through an enforced isolationism, but precisely through complete immersion in the dense texture that arises by decomposition into pieces, careful choices made locally, and the resulting appreciation of the need to *make explicit the most minute of links between parts of a whole*, as one gradually assembles a more global or unified perspective. In this connection, we could also mention one of his descriptions of his famous "contrapuntal radio" programs from the 70's, in which he claimed to try "to have situations arise cogently from within the framework of the program in which two or three voices could be overlapped, in which they would be heard talking—simultaneously, but from different points of view—about the same subject," a chief aim of which was supposedly to "test the degree to which one can listen simultaneously to more than one conversation or vocal impression" ("The Idea of North: An Introduction," in Page, *Glenn Gould Reader*).

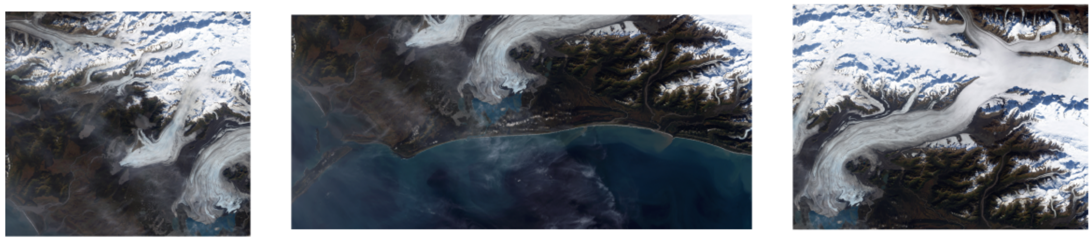
reducing the individual part (voice, line, note) to its participation in a prefabricated idea of totality and relying on a dubious notion of some "immediate" continuity, montage/splicing (like the sheaf construction) is fundamentally *inductive*, allowing the individual component materials of a work to create their own formal structure piece by piece via its insistence on the transparent and explicit unfolding of the principles and translation formulas by which the component parts can be patched together locally. According to Gould, it is precisely through the initial discontinuity induced by the decomposition into parts and cutting process in montage/splicing that the task of making explicit the principle of their reorganization/patching into a unified totality is allowed to emerge, and is no longer regarded as something *a priori* or to be taken for granted. Just as in the sheaf construction, this approach essentially involves both cutting (decomposition/discontinuity) and local patching or gluing (recomposition/continuity) of sections that, together, gradually cover the entire piece.

For a more concrete (if very rough and simplified) idea of how the splicing/montage approach to recording amounts to the construction of a sheaf, consider a musical score consisting of 32 measures. We might then consider that the "space" of the entire score has been decomposed into three principal parts or pieces: (A) spanning from measure 1 to the end of measure 16; (B) spanning from the beginning of measure 8 until the end of measure 24; and (C) spanning from the beginning of measure 16 until the final measure. Together, these portions obviously collectively cover the entire 32-measure score, and there are the obvious overlapping measures (and the induced inclusions). We can now imagine that to each section (A)-(C), there corresponds a (possibly very large) set of distinct recordings. If, for some selection of individual recordings from each of the three regions (A)-(C), the selected recordings can be made to agree on their overlap—via some system of translation functions, e.g., slowing down one recording to match the tempo of another—then they can be spliced together into a unique recording of the entire work.

Example 6.0.11. Detectives collect certain information pertaining to a crime that purportedly occurred in some area during a certain time interval. This information will most likely be heterogenous in nature, i.e., they may have camera footage of some part of the scene, some eyewitness testimony, some roughly time-stamped physical data, etc. These various pieces of data are all considered to be local in the sense that they are assumed to concern, or be essentially indexed to, a certain limited region of space-time, e.g., camera footage of one of the parking lot's exits that is literally time-stamped or an eyewitness who claims to have heard a scream coming from the southern end of the parking lot sometime between 8 pm and 8:30 pm. In terms of the underlying space-time regions to which these various pieces of information correspond, the various pieces of information may very well have to be checked for overlap, e.g., an eyewitness's testimony with respect to a half-hour interval might be checked against the camera feed concerning that same time interval (and concerning the same area). In general, the various pieces of data over the same interval may corroborate one another or contradict one another, either entirely or in some particular respect or with respect to some sub-region of their overlap. It is not as simple as verifying whether or not they provide the *same* information. It may happen, for instance, that the parking lot is constructed in such a way that certain barriers acoustically account for why the witness heard the scream coming from the southern end of the parking lot, when in fact it could only have come from the western end (which is where the camera shows the victim in conflict during that time). It is the job of the detective to find the appropriate translation functions making sense of these at first (potentially) conflicting local pieces of data and then use these functions to "glue" together, step by step, the data that can be made to locally cohere into a self-consistent account of what occurred over the entire spatio-temporal interval in question. In a sense, given a presheaf assigning information (camera data, propositions, etc.) locally over some collection of space-time regions, the detective is looking, in trying to solve the case, to make a sheaf over the entire space-time interval covered by all those regions.

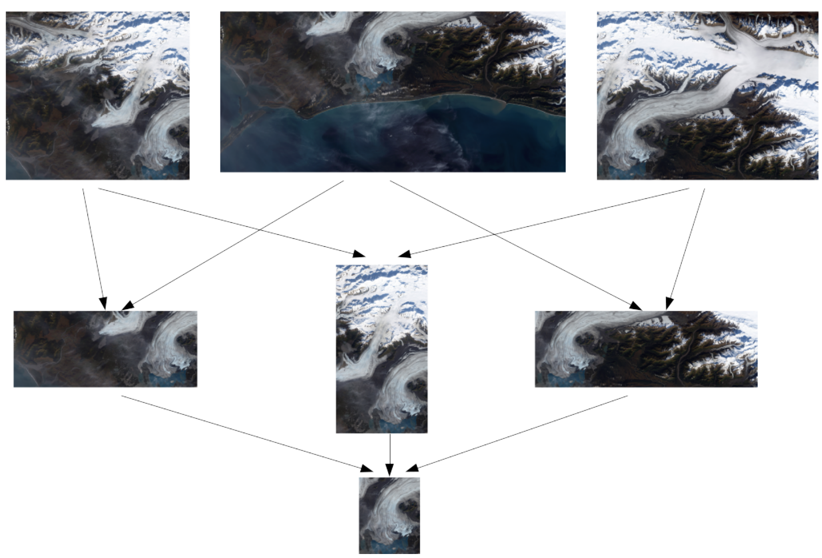
Example 6.0.12. For the next example, we consider a satellite, or various satellites, making passes over portions of earth, collecting data as it goes. For concreteness, consider some specific portion of the earth, say Alaska, or that part of Alaska where the Bering Glacier lies, as a topological space X. Then given an open subset $U \subseteq X$, we can let $S(U)$ denote the set of functions from U to C, where C might be the set interval of wavelengths in the light spectrum, or some geo-referenced (perhaps timestamped) intensity-valued image data, or some other data corresponding to the data feed of the satellites (or the processing thereof). This presheaf S is in fact a sheaf, since we can indeed fuse together the different data given over the open sets of X, forming a larger patched-together image of the glacier. For concreteness, assume we are given the following selection of three satellite *images* of the Bering Glacier, chosen from among the possible very large sets of images assigned to each region:[29]

[29] The images come from the satellite Landsat 8, found here: https://earthobservatory.nasa.gov/IOTD/view.php?id=4710.



v_1 v_2 v_3

Each of the $v_i \in S(U_i)$ correspond to value assignments throughout or over certain subsets, U_1, U_2, U_3 of X, which together cover some subset $U \subseteq X$—say the region of Alaska corresponding to the glacier. In terms of the data "sitting over" each of these regions, as provided by each of the satellites in the form of individual images, we can notice that the restriction of v_1 to the region $U_1 \cap U_2$ is equal to the restriction of v_2 to the same subset $U_1 \cap U_2$, and so on, all the way down to their common restriction to $U_1 \cap U_2 \cap U_3$:

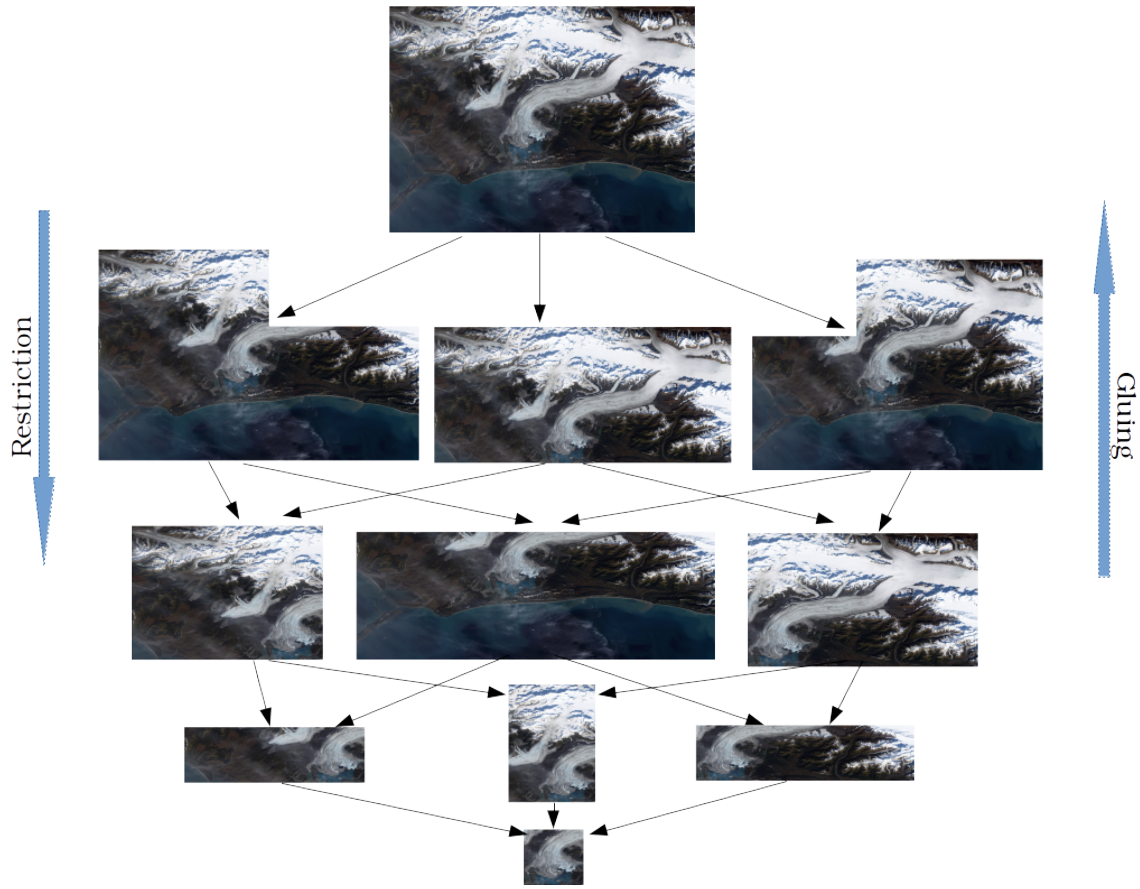


One can thus immediately *see* that the sheaf condition is met, which means that we can in fact patch together the given local pieces or sections over the members of the open covering of U to obtain a section over all of $U = U_1 \cup U_2 \cup U_3$. In summary, we have the following inclusion diagram (on the left) describing the underlying topology, paired with the sheaf diagram (on the right) with its corresponding restriction maps (notice the change in direction):

$$\begin{tikzcd}
 & U & \\
U_1 \cup U_2 \arrow[ur] & U_1 \cup U_3 \arrow[u] & U_2 \cup U_3 \arrow[ul] \\
U_1 \arrow[u] \arrow[ur] & U_2 \arrow[ul] \arrow[ur] & U_3 \arrow[ul] \arrow[u] \\
U_1 \cap U_2 \arrow[u] \arrow[ur] & U_1 \cap U_3 \arrow[ul] \arrow[ur] & U_2 \cap U_3 \arrow[ul] \arrow[u] \\
 & U_1 \cap U_2 \cap U_3 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & S(U) \arrow[dl] \arrow[d] \arrow[dr] & \\
S(U_1 \cup U_2) \arrow[d] \arrow[dr] & S(U_1 \cup U_3) \arrow[dl] \arrow[dr] & S(U_2 \cup U_3) \arrow[dl] \arrow[d] \\
S(U_1) \arrow[d] \arrow[dr] & S(U_2) \arrow[dl] \arrow[dr] & S(U_3) \arrow[dl] \arrow[d] \\
S(U_1 \cap U_2) \arrow[dr] & S(U_1 \cap U_3) \arrow[d] & S(U_2 \cap U_3) \arrow[dl] \\
 & S(U_1 \cap U_2 \cap U_3) &
\end{tikzcd}$$

$$\mathscr{O}(X)^{op} \xrightarrow{\quad S \quad} \mathbf{Set}$$

In terms of the actual images, the sheaf diagram on the right is pictured below, where we can think of the restriction maps as performing a sort of "cropping" operation, corresponding to a reduction in the size of the domain of the sensor, while the gluing operation corresponds to patching or gluing the images together along their overlaps all the way up to the topmost image (which of course corresponds to the section or assignment over all of U).



This mosaic example gives a particularly concrete illustration and motivation for an alternative definition of a sheaf, namely as a presheaf $F : \mathscr{O}(X)^{op} \to \mathbf{Set}$ that moreover *preserves limits*—where, because we use the opposite category for domain in defining the presheaf, this means that *colimits* get sent to limits in **Set**. We can see that in the lattice of open subsets of X, for an I-indexed family of open subsets $U_i \subset U$ (in the particular case described above, $I = 3$) that *covers* U—in the exact sense that the entire diagram comprised of the sets U_i and the inclusions of their pairwise intersections $U_i \cap U_j$ has U for its colimit—the contravariant functor S given above *preserves* this colimit in the sense that it sends it to a *limit* in **Set**. In terms of the universal characterization of these notions developed earlier, one can basically immediately see that while all arrows "fall into" U (think of U as the nadir of a cone), any other possible object in this poset will have to pass through U, i.e., U is initial; likewise, it is patently visible that the summit $S(U)$ will be terminal among cones.

Thus far, we have confined our attention to sheaves F on a topological space X. For such sheaves on spaces, there are basically two candidate descriptions of a sheaf: (i) the "restriction-collation" description, which we have begun to describe; and (ii) the approach that takes a sheaf to be a *rule* assigning to each point x of the underlying space a set F_x of the "germs" at x of the functions being considered (where these "germs" are basically equivalence classes identifying sets that look "locally the same" around x), after which these sets F_x are then "pasted" together by a suitable topology to form a space (or bundle) projected onto X (a suitable function for this sheaf then being a "cross section" of the projection of this bundle), according to which perspective the sheaf F can be thought of as a set F_x "varying" with the points x of the space. Focusing on the restriction-collation description, consider the special case of continuous functions and how a topology on a set X serves to define the continuous functions there. For instance, we have the continuous functions from the space X to the reals $\mathbb{R}$, or from any open set U in X to $\mathbb{R}$. Whether or not a function $f : U \to \mathbb{R}$ is continuous is something that can be *determined locally*, where this amounts to saying

1. **Restriction (or Identity)**: If $f : U \to \mathbb{R}$ is continuous, and $V \subset U$ is open, then restricting f to V, i.e., $f|_V : V \to \mathbb{R}$, yields a continuous function as well.
2. **Uniquely collatable (or Gluability)**: If U is covered by open sets U_i, and the functions $f_i : U_i \to \mathbb{R}$ are continuous for all $i \in I$, then there will be *at most one* continuous $f : U \to \mathbb{R}$ with restrictions $f|_{U_i} = f_i$ for all i. Furthermore, this f will

exist iff the given f_i *match* on all the overlaps $U_i \cap U_j$ for all i, j, i.e., $f_i x = f_j x$ for all $x \in U_i \cap U_j$.[30]

One might alternatively think about the "localness" described thus as involving two sorts of compatibility conditions or as constraints tending in two directions (the first "downward" and the second "upward"): (1) those that require that information over a larger set is compatible whenever *restricted* to information over a smaller open set; (2) those that involve the *assembly* of information on smaller opens into information over larger open sets. In this manner, one might also think of the first condition as the "localizing" part of the sheaf construction, and the second condition as the "gluing" or "globalizing" part. While continuous functions provide a particularly natural example of these sorts of requirements, there is certainly no need to restrict ourselves to *continuous* functions; many other structures on a space X are in fact "determined locally" in much the same sense as the above, thus permitting the sheaf construction for a wide class of structures or collections of functions (including things that are only "function-like").

But even when structures are determined locally, sometimes local properties alone do not suffice to determine global properties. In such cases, we will not have a sheaf. In general, a presheaf can fail to be a sheaf in two (independent) ways:

- **Non-locality**: If a presheaf has a section $s \in F(U)$ that cannot be constructed from sections over smaller open sets in U—via a cover, for instance—then F fails to be a sheaf.

[30]This perspective of restriction-collation comes from MacLane and Moerdijk, *Sheaves in Geometry and Logic*.

- **Inconsistency**: If a presheaf has a pair of sections $s \neq t \in F(U)$ such that when restricted to every smaller open set they define the same section, then F fails to be a sheaf. In other words, informally, the presheaf has local sections that "ought to" patch together to give a unique global section, but do not.

However, in general, there is a standard procedure for completing a presheaf to make it a sheaf. Since there are two fundamental ways a presheaf can fail to be a sheaf, this process, usually dubbed "sheafification," can be thought of as basically doing one of two things: (1) it *discards* those extra sections that make the presheaf fail to satisfy the locality condition; (2) it *adds* those missing sections which, had they been present, would allow the local sections to glue together into a unique global section, satisfying the gluability condition. In other words, with respect to the second of these two, we are adding functions to the global set that restrict to compatible functions on each of the opens, and then, recursively, we continue adding the restrictions of the newly-generated global functions.

Philosophical Pass (4th): Continuity and Generality in Local-Global Passage

A sheaf is not to be situated in either the local (restriction) or the global (collation) registers, but rather is to be located in the passage forged between these two, in the translation system or glue that mediates between the two registers. The transit from the local to the global secured via the sheaf gluing (collatability) condition provides a deep but also precisely controllable connection between continuity (via the emerging system of transition or translation functions guaranteeing coherence or compatibility between the local sections) and generality (global sections). By separating something into parts, i.e., by specifying information locally, considering coverings of the relevant region, and enabling the decomposition or refinement of value assignments into assignments over restricted parts of the overall region (restriction condition), we are presented with a problem, a problem that in a sense can only first appear with such a “downward” movement towards greater refinement. Without having separated something into parts, we may appear to have a sort of trivial or default cohesion of parts, where, without being recognized in their separation, the parts yet remain implicit and so the glue binding them together or the rule allowing one to transit from one part to another in a controlled fashion is simply not visible. However, having decomposed or discretized something into parts, we are at once presented with this separation of parts and the problem of finding and making explicit the glue that will serve to bind them together. On this perspective, a sheaf is neither a purely continuous nor a purely discrete structure. It is a way of taking information that is locally defined or assigned and decomposing those assignments in a controlled fashion into assignments over smaller regions so as to draw out the specific manner of effecting translations or gluings that obtain between those particular assignments with respect to their overlapping regions, and then using this now explicit system of gluings to build up a unique and comprehensive value assignment over the entire network of regions. In this sense, a sheaf equally involves both (i) controlled decomposition (discreteness), and (ii) the recomposition (continuity) of what is partial into an architecture that makes explicit the special form of cooperation and harmony that exists between the decomposed items, items that may have previously been detached, or which may have only *appeared* to “stick together” because we had not bothered to look closely enough.

Via the restriction/localization step, sheaf theory teaches us that we do not command a more global or integrated vision by renouncing the local nature of information or distinct planes and textures of reality or by glossing over the minute passages between things. Instead, it forces us to first become masters of the smallest link and, *precisely through that control* of the passages between the local parts, forge a coherent ("collatable") vision of the largest scope.

Phenomenologically speaking, data or observations are frequently presented to us in "zones," "fragmented" or isolated in some way. These items can be thought of as various light-beams (perhaps of specific hues or brightness) cast over (and covering only parts of) a vast landscape, some of which may overlap. Even if this data clearly emerges as *evolving* over some region, it remains indexed or determined in some way by a particular "zone" or context. One interpretation of this initial "particularity" would be to suggest that the very fact that certain information initially presents itself in this local and bounded fashion is an indication that we are dealing with various discrete approximations, presented piecemeal, to phenomena that may in fact "really" be continuous. Whether or not that is the case, it is sure that in its presentation to us in fragmented form, this step in the process is closely allied with the discrete. For centuries, the modes of restoring continuity to such partial information have been more or less haphazard. A sheaf removes this aspect of haphazardness. Significant is the at once *progressive* and *necessary* nature of the sheaf concept: how by gradually (progressively) covering fragments of reality, and then systematically gluing them together into unique global solutions (necessary), the construction of sheaves encourages us to shift away from our standard ontologies or descriptions of reality as anchored in some "absolute" towards a more "contrapuntal and synthetic" perspective capable of registering "relative universals."[31]

[31]In Zalamea, *Synthetic Philosophy of Contemporary Mathematics*, the "progressive-necessary" feature of sheaves, the importance of the analysis-synthesis polarity, and the more general shift (in category theory)

With the sheaf construction, a global vision is not *imposed* on the local pieces, obliterating the local nature of the presented information via some "sham" generalization, but emerges progressively, step-by-step, through the unfolding of precise translation systems guaranteeing the compatibility of the various components. A sheaf does not attempt to suppress the richness and polyphony of data in its particularity and relative autonomy, coercing a kind of standardized agreement as so many past models of generality (universality) have done. A sheaf is like a master composer who is not content to have her harmony prefabricated for her by habitual associations, or who would achieve harmony only at the expense of suppressing all contrapuntal impulses and polyphony, imposing it "from above," or restraining the local freedom of each voice to roam with some independence from the constraints that bind it in the name of some prefabricated schema; rather, the sheaf composer achieves harmony only progressively, first by letting each component part unfold, in its relative autonomy, its own laws, then by insisting on making explicit even the most minute of links and transits between the laws of movement of each of the parts, securing locally smooth passages for each transition, and from the glue or constraints that emerge out of this process, begins to build up a larger ensemble, step by step. It is not a compromise between the local and the global in the name of some idealogical preference for the more global or universal. Sheaves earn their place as true mediators by virtue of their complete realization of the idea that—to paraphrase Hegel—true mediation comes about only from preserving the extremes as such, and true universality comes about only by sinking as deeply as possible into the particular.

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away from ontologies anchored in the "absolute" are emphasized. See chapters 8 and 9, in particular, for discussion of these things.

Philosophical Pass (5th): Sheaf Cohomology and Return of the Discrete

If sheaves represent local data, sheaf cohomology is a tool used for extracting global information from local data and for systematically detecting, representing, measuring, and relating *obstructions* to the extension of the local to the global (or, equivalently, to the formation of a compatible patchwork of local systems of information). Via cohomology, global compatibilities between pieces of local data can display global qualitative features of the underlying topology of the data structure. Via sheaf (co)homology, we can, for instance, isolate potential "holes" in data collections, or find information feedback loops that might result in faulty inferences (for instance, via facebook feeds).[32]

If the sheaf compatibility conditions require controlled transitions from one local description to another, enabling progressive patching of information over overlapping regions until a unique value assignment emerges over the entire region, higher (non-vanishing) cohomology groups basically detect and summarize (in an algebraic fashion) obstructions to such local patching and consistency relations among various dimensional subsystems. In other words, it can be thought of as measuring (for some cover) how many incompatible (purely local) systems we have to throw out in order to be left with only the compatible systems. Such non-vanishing groups give an algebraic representation of something like the *resistance* of certain information (assigned to a part of a space) to integration into a more global system. In this respect, sheaf cohomology could intuitively be thought of as capturing the non-globalizability or non-extendibility of a given information structure in relation

[32]In another context, Abramsky (see Abramsky and Brandenburger, "The Sheaf-Theoretic Structure Of Non-Locality and Contextuality") develops the idea that obstructions to the existence of global sections correspond to *contextuality*.

to other overlapping data structures, thereby possibly indicating a (sub)system's degree of contextuality. Both in its algebraic representation and in this general interpretation, then, such non-vanishing cohomology groups might be thought of as giving us a picture of just how "non-integrated" a system of information over a space may be. On the other hand, vanishing cohomology groups indicate the mutual compatibility or 'globalizability' of local information systems (they supply the global sections). In this way, sheaf cohomology emerges as a tool for representing (algebraically) what might be thought of as the degree of generality (or particularity) of a given system of measurement or interlocking ways of assigning information to a space. In short, if the collation condition in the sheaf construction aligns them with continuity in the sense that it ensures smooth passage from the local to the global, sheaf cohomology is something like its discrete counterpart providing us an algebraic summarization of when and how such local-global passages might be blocked.

Some years before the invention of sheaf theory, Charles Peirce argued that "continuity is shown by the logic of relations to be nothing but a higher type of that which we know as generality. It is relational generality."[33] Such suggestive, if somewhat cryptic, remarks provoke us to take a closer look at the connections between generality and continuity that emerge in the context of sheaves. We know that a sheaf enables a collection of local sections to be patched together uniquely given that they agree (or that there exists a translation system for making them agree) on the intersections. Consider the satellite sheaf above. Recall the way in which the sheaf (collation/gluing) condition ensures a systematic passage from local sections (images of parts of the glacier) to a unique global section (the image of the

[33]Peirce, *Collected Papers of Charles Sanders Peirce*, 6.190.

entire glacier). Where the localizing step of the sheaf construction might be thought of as analytic, decomposing an object into a multitude of individual parts (local), the gluing steps are synthetic in restoring systematic relations between those parts and thereby securing a unique assignment over the entire space (global). The global section should not be thought of as a single (topmost) image, but rather as the entire network of component parts welded together via compatibility relations. In this sense, generality can be understood in terms of the systematic passages from the local to the *global*, a passage that is strictly *relational*, in that the action of the component restriction maps is precisely an enforcing of certain relations or mutual constraints between the local sections (that are then built up, along the lines of these relations, into a global section). One might further think of the indexing (domain) category in the (pre)sheaf construction as providing the context specifying the possible scope of the generality—just *how* global the global section is—of a given sheaf diagram. In this way, the degree of generality achieved by a particular sheaf construction can be thought to depend upon the form of the indexing category, thereby *relativizing* the concept of generality. In so far as such distinct systems for the production of (relative) generalities can be themselves compared via natural transformations, one might also think of this as introducing yet another (higher-order) layer of relationality and relativity into the notion of generality. In these ways, via the sheaf concept, Peirce's suggestive idea that

$$\text{Continuity} = \text{Relational Generality}$$

is thus given a particularly powerful interpretation.[34]

In the next section, I discuss some final contributions to the continuity-generality connection in the context of *toposes* and dialectics, before concluding with a final section summarizing the main "philosophical" take-aways of this chapter.

Section 4: A "Dialectic" between Continuity and Discreteness?

Toposes

The definition of a presheaf can easily be generalized beyond the topological case to an arbitrary (small) category **C**; moreover, the concept of a sheaf, as presented thus far in terms of the usual topological coverings, restriction and collation, can be extended beyond the usual topological spaces, admitting a definition on more general "topologies." This leads to the notion of a *Grothendieck topology*, which is based on a more general concept of a *covering*, which can be defined for any category. The basic idea is that a cover should represent something like the spatial interconnection between the parts of a structure. For example, the set of substrings {"*Groth*", "*thend*", "*ndieck*"} can be thought to cover the string "*Grothendieck*" because we can glue together the substrings along the overlapping parts ("th" and "nd") to yield the original string. Via this wider notion, we can make use of topological intuition in situations where there does not appear to be any topology (in the traditional sense) at play.

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[34]It is perhaps worth noting that while Peirce's suggestive remarks such as "Every General is a continuum vaguely defined" have partly inspired some of the connections I have drawn between continuity and generality, there are a number of serious differences between how (and on what grounds) Peirce develops these connections and what I believe applies to sheaf theory. Peirce's views on continuity evolved significantly throughout his career, though, so I do not make any claims to his final view on the matter; however, with respect to certain of his definitions, the divergence from the sheaf perspective (as I have been developing it) is not insignificant. In the interests of space, however, I have decided not to include further discussion of these matters.

One way of developing this revised notion of "covering" is by considering how the construction can be accomplished in any category $\mathbf{C}$ assumed to have pullbacks, namely by considering the indexed families, for a given object c of the category $\mathbf{C}$, of maps to c, and then building coverings out of the set of such indexed families. We can then just repeat the classical (topological) definition of a sheaf. However, to develop the fully general notion of a Grothendieck topology, we need the more general notion of a *sieve*. Given an object c in the category $\mathbf{C}$, a *sieve* S on c (or a "crible" on c) is a family S of arrows in $\mathbf{C}$ all of which have codomain c such that if $f \in S$, i.e., $f : b \to c$, and for some $g : a \to b$ the composite $f \circ g$ is defined, then $f \circ g$ is also in S. In other words, a sieve S on c is a collection of arrows with codomain c closed under composition on the right. This definition means that any path to some other object b followed by a path from b to c itself is a path in S. Moreover, we require that if S is a sieve on c and $h : d \to c$ is any arrow to c, then we have that

$$h^*(S) = \{g \mid \mathrm{codomain}(g) = d, (h \circ g) \in S\} \tag{6.5}$$

is a sieve on d. A generic picture of a sieve might look something like:

$$\begin{tikzcd}
& \cdots & & & \cdots & & & \\
a' \arrow[drr] \arrow[ddrrrr, red, bend right] & & a'' \arrow[d] \arrow[ddrr, red] & b' \arrow[dr] \arrow[ddr, red] & & b'' \arrow[dl] \arrow[ddl, red] & & d' \arrow[dl] \arrow[ddlll, red, bend left] \\
& & a \arrow[drr] & & b \arrow[d] & & d \arrow[dll] & \\
& & & & c & & &
\end{tikzcd}$$

In slogan form (explaining the term "sieve"):

> If b "goes through" the sieve (i.e., if there is an arrow $b \to c$), then so too does anything "smaller" (i.e., whenever there is an arrow $b' \to b$, then the arrow $b' \to c$ goes through as well).

The above definition takes a sieve to be a *set* of arrows satisfying certain constraints; as such, it is less category-theoretic than we might like. But alternative characterizations of sieves can be given, yielding three representations in total: a sieve can be represented as a set, a sub-category of the comma category, or a sub-functor of the representable functor. To define a Grothendieck topology, we need only establish one last piece of terminology: for an object c of $\mathbf{C}$, the set $t_c = \{f \mid \text{ codomain}(f) = c\}$ of *all* arrows into c will be a sieve, called the *maximal sieve* on c. We are now in a position to define the following:

Definition 6.0.11. A *Grothendieck topology* (or *localization system*) on a category $\mathbf{C}$ is a function J that assigns to each object c of $\mathbf{C}$ a collection $J(c)$ of sieves on c, in such a way that

- (i) *identity cover (or maximality axiom)*: the maximal sieve t_c is in $J(c)$;
- (ii) *stability under change of base (or stability axiom)*: if $S \in J(c)$, then $h^*(S) \in J(d)$ for any arrow $h : d \to c$;
- (iii) *stability under refinement (or transitivity axiom)*: if $S \in J(c)$ and R is any sieve on c such that $h^*(R) \in J(d)$ for all $h : d \to c$ in S, then $R \in J(c)$.

Informally, a Grothendieck topology can be thought of as ensuring that, in regarding a given object c, we regard it by decomposing it into all the perspectives or "eyes" that fall on it.

If $S \in J(c)$, we say that S is a *covering sieve*, or that S *covers* c. We will also say that a sieve S on c *covers an arrow* $f : d \to c$ if $f^*(S)$ covers d. In other words, S covers c iff S covers the identity arrow on c. In these terms, the axioms for a Grothendieck topology can equivalently be formulated as follows (this is called the *arrow form* of the definition):

- (ia) *identity*: if S is a sieve on c and f is in S, then S covers f; informally, this says that any "open set" covers itself, or (in terms of sets) that any set is covered by all its possible subsets.
- (iia) *stability*: if S covers an arrow $f : d \to c$, it also covers the composition $f \circ g$, for any arrow $g : e \to d$; informally, this says that coverings "pullback" (i.e., closure under fibered products, generalizing the usual notion that the intersection of two open sets is open).
- (iiia) *transitivity*: if S covers an arrow $f : d \to c$, and R is a sieve on c which covers all arrows of S, then R covers f; informally, this says that a cover of a cover will be a cover.

It further follows from the axioms that any two covers have a common *refinement*, i.e.,

- (iv) if $R, S \in J(c)$, then $R \cap S \in J(c)$,

or in arrow form:

- (iva) if R and S both cover $g : d \to c$, then $R \cap S$ covers g.

This rather flexible notion is often said to be a vast generalization of the topological space notion. Many examples could be given, but let me just mention one: if $\mathbf{C}$ is any category, the *minimal topology* (also called the *trivial topology*) T on $\mathbf{C}$ is the one in which the only sieve covering an object c is the maximal sieve t_c. This topology is the coarsest among all topologies on $\mathbf{C}$.[35]

In the classical definition of a sheaf on a topological space X, sheaves were basically an association of information to the open sets of X, satisfying a gluing axiom specified in

[35]For this topology, the reader should try to convince themselves that every presheaf is automatically a sheaf, anticipating the fact that the category $\mathbf{Sh}(\mathbf{C}, T)$ is none other than the presheaf category $\mathbf{Set}^{\mathbf{C}^{op}}$.

terms of a *pointwise covering*, i.e., for $U \subset X$, we have that open subsets $\{U_i\}_{i \in I}$ *cover* U iff $\bigcup_i U_i = U$, and where every point that is in U comes from some U_i. Sheaf theory investigates the global consequences of properties that are defined locally; in the classical definition of a sheaf, this notion of *local* is specified via the topology. With Grothendieck's generalization of the notion of topology to any category, we can develop a more encompassing notion of a sheaf on any small category, specifying a localizing system via the more general notion of covering, thereby disposing of the condition that the base be formed of open sets and the lattice formed by inclusions arrows between these sets. In this manner, when considering an arbitrary category, the *points* vanish, leaving only the "open sets," related no longer by inclusion arrows but by arbitrary arrows. The "topology" is then entirely captured by the specification of the "cover."

The basic motivating idea here is that there is no need to restrict ourselves to topological spaces to do sheaf theory. As long as the category gives information similar to open covers, we can generalize beyond the category of open sets of a topological space. A Grothendieck topology emerges as basically a rule for specifying when certain objects of a category "ought to" cover another object of the category, "purifying" (by axiomatizing) the usual topological notion of an open cover. A Grothendieck topology puts a structure on a category that makes the objects of that category "behave like" the open sets of a topological space. By developing these notions, the important notion of a *site* emerges as a category together with a choice of Grothendieck topology. More precisely, a *site* is a pair $(\mathbf{C}, J)$ consisting of a small category $\mathbf{C}$ and a Grothendieck topology J assigned to $\mathbf{C}$. Sheaves on a category can then be defined in more general terms, and more generally still, the sheaves on a site $(\mathbf{C}, J)$ themselves form a category, where the maps are the natural transformations (between presheaves). This category of sheaves, which we denote as $\mathbf{Sh}(\mathbf{C}, J)$, forms a full subcategory of the functor category $\mathbf{Sets}^{\mathbf{C}^{op}}$. This in turn enables us to define the all-important notion of a *Grothendieck topos* as the category of sheaves on a site.

Definition 6.0.12. A *Grothendieck topos* is a category which is equivalent to the category $\mathbf{Sh}(\mathbf{C}, J)$ of sheaves on some site $(\mathbf{C}, J)$.[36]

Example 6.0.13. If we let T be the trivial topology, then given a small category $\mathbf{C}$ with the trivial topology on it, the T-sheaves on $\mathbf{C}$ are just the presheaves on $\mathbf{C}$. Thus, $\mathbf{Sh}(\mathbf{C}, T)$ reduces to the presheaf category $\mathbf{Set}^{\mathbf{C}^{op}}$.

In general, one can think of the category of sheaves over a space (following Grothendieck) as something like a superstructure of interlocking measurement systems that captures what is most essential about a space. The definition of a Grothendieck topos can be applied to any category of sheaves associated to a site, and in particular this notion of topos encompasses any presheaf (or variable set) category $\mathbf{Set}^{\mathbf{C}^{op}}$ (just set J as the minimal topology). While quite general, then, in particular this includes $\mathbf{Set}^{\mathbf{1}^{op}} = \mathbf{Set}$, i.e., sheaves on the one-point topological space. The notion of a Grothendieck topos in fact depends on the assumed model of set theory. Lawvere and Tierney introduced the notion of an *elementary topos* in part to provide a characterization of categories that resemble Grothendieck toposes (from one perspective) but which can be defined strictly by elementary axioms that are independent of set theory. An elementary topos is a category $\mathscr{E}$ that

- (i) has all finite limits (i.e., has pullbacks and a terminal object);

[36]For more details on the construction of sheaves on a site and Grothendieck toposes, see MacLane and Moerdijk, *Sheaves in Geometry and Logic*, III.4.

- (ii) has exponentials (i.e., is cartesian closed); and
- (iii) has a subobject classifier satisfying certain conditions.[37]

Every Grothendieck topos is an *elementary topos* but an elementary topos need not be a Grothendieck topos.[38]

Philosophical Pass (6th): Toposes and the Concept of Space

As Grothendieck himself notes,[39] the concept of a Grothendieck topos in a sense joins together the continuous and structures that would appear to be thoroughly algebraic and discrete. One might compare this sort of argument to Grothendieck's notion (one he traces back to Riemann) that, contrary to the dominant approach that says that we are always "approximating" (allegedly) continuous phenomena with our discretizations,

> It could well be that the ultimate structure of space is discrete, while the continuous representations that we make of it constitute perhaps a simplification (perhaps excessive, in the long run...) of a more complex reality; that for the human mind, "the continuous" was easier to grasp than the "discontinuous", and that it serves us, therefore as an "approximation" to the apprehension of the discontinuous. This is a remark of a surprising penetration in the mouth of a mathematician [namely, Riemann], at a time when the Euclidean model of physical space had never yet been questioned; in the strictly logical sense, it was rather the discontinuous which traditionally served as a mode of technical approach to the continuous.[40]

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[37]Details of all this can be found in any standard text on topos theory, such as Johnstone, *Topos Theory*.

[38]Some of these are useful in the study of higher-order intuitionistic type theory; in practice, one important difference between elementary toposes and Grothendieck toposes is that the presence of sites in Grothendieck toposes provides a setting where one can more readily use geometric intuitions in the study of a topos. For more details on elementary topoes and the Lawvere-Tierney topology used in their definition, see MacLane and Moerdijk, *Sheaves in Geometry and Logic*, V.1.

[39]See, in particular Grothendieck, "Recoltes et Semailles: Reflexions et temoignage sur un passe de mathematicien," Promenade 13.

[40]Ibid.

In a basic sense, toposes can be seen to be unifying in their ability to "house" disparate mathematical constructions. Moreover, toposes are useful not only for studying relationships between different mathematical theories, but also for examining a given mathematical theory from a variety of different points of view. Most importantly, however, the notion of a topos emerges as a solution to the problem of providing a truly unified treatment of the discrete and the continuous. In the notion of a topos, the (discrete) combinatorial diagrammatic approach (via categories) is paired harmoniously with the usual (continuous) concepts native to topological spaces. The concept of a site, moreover, realizes in a particularly powerful and general fashion the idea that one can examine a space without looking at its *points*. Via the notion of a topos, we do not focus on a space itself and its points, but on how that space can be made to define a variable structure that varies over the points of the space. Much has been said about how in attempting to obtain a topos appropriate for a particular sort of math by constructing the domain of variation (via a site) and then considering the category of sheaves over that site, this gives rise to the perspective of *variable set theory*, wherein the standard constant universe of sets is replaced by something like a plurality of "possible worlds."[41] In this connection, we note that classical logic (with respect to first-order

[41] As Johnstone claims in Johnstone, *Topos Theory*, xvii, this could in fact even be regarded as the very essence of the topos-theoretic view of things:

> it consists in the rejection of the idea that there is a fixed universe of "constant" sets within which mathematics can and should be developed, and the recognition that the notion of "variable structure" may be more conveniently handled within a universe of *continuously variable* sets than by the method, traditional since the rise of abstract set theory, of considering separately a domain of variation (i.e. a topological space) and a succession of constant structures attached to the points of this domain. In the words of Lawvere, "Every notion of constancy is relative, being derived perceptually or conceptually as a limiting case of variation, and the undisputed value of such notions in clarifying variation is always limited by that origin."

structures) can be shown to be a *limit* of intuitionistic logic, the "true home" of sheaves. Not only is classical logic merely an extreme limit in a vast sea of (weaker) intermediate logics, but sheaves on a locale give rise to a generalized set-theory, i.e., a set-theory that supports intermediate truth values. The logic of sheaves in general is intuitionistic, i.e., the law of the excluded middle fails. In all of these respects, there is a move away from the discrete, "point-based" and classical perspectives; moreover, the failure of the excluded middle in general (but its applicability for closed formulas, i.e., individuals, in certain topos models) suggests deep connections with generality. The close connection between intuitionistic logic and sheaves provides another important perspective on the connections between continuity and generality, connections already anticipated by Charles Peirce's suggestion that what makes something general is ultimately due to the failure of application of the law of the excluded middle.[42]

In general, a topology is designed to capture and understand two things: locally-defined phenomena and continuous transformations. If Grothendieck's notion of a site supplies us with a more general notion of a localization system, consideration of the sheaves on a site provides a continuous perspective on a category that may in principle be quite "non-topological" or "non-space-like." The notion of *geometric morphisms* between toposes extends the generalized notion of continuous transformations even further.

The main purpose of this section is to build to a discussion of *cohesive toposes*, and to look at how the relations of continuity and discreteness are framed in this context. This

[42]See, e.g., Peirce, *Collected Papers of Charles Sanders Peirce*, 5.448.

will also provide us with the opportunity to discuss what it means to say that there is a *dialectic* between continuity and discreteness. We typically think of "cohesion" in terms of topology, where cohesion is something like the specification of how points or objects in a space "hang together." But many different contexts (categories) present us with differing modes of cohesion and variation. Since functors sometimes allow us to relate and compare categories, the question of which settings for modeling objects are more cohesive/variable and which are more discrete/constant should involve functorial comparisons. One would like to more precisely study and control the contrast between the degree and type of continuity (*qua* cohesiveness/variability) or discreteness (*qua* non-cohesion/constancy) of these related but distinct settings. Geometric morphisms between toposes enable us to begin to compare the manner in which objects living in different environments (categories, toposes) hang together or cohere—giving rise to a "science of cohesive toposes." In order to build up to cohesive toposes, I must briefly introduce geometric morphisms.

Geometric Morphisms

While we have briefly looked at the notion of a topos, I have not yet defined an appropriate notion of *morphisms of toposes*. Using the notion of an elementary topos, one might suspect that this could be captured by a functor that preserves finite limits, exponentials, and the subobject classifier. This indeed defines a legitimate functor between toposes, namely a *logical functor*. While such functors can play an important role in the theory—in particular, being of use to one who has adopted the perspective of an elementary topos as the syntactic category of a higher-order (intuitionistic) type theory—there is another natural type of morphism to consider between toposes: *geometric morphisms*. In certain ways, the notion of a geometric morphism can even be regarded as the more vital of the two. Roughly, in the context of the perspective that regards toposes as "generalized spaces," geometric morphisms can be regarded as "generalized continuous maps." As such, such morphisms can initially be thought of as preserving the geometric structure of toposes (this is to be compared to how logical morphisms can be thought of as preserving the elementary logical structure). To properly describe geometric morphisms, we first need to discuss *adjunctions*.

Adjunctions via Galois Connections

The notion of adjunction applies when we are interested not so much in a relation (or isomorphism) between two categories but in the relation between specific functors between those categories. If we restrict our attention to the category **Pos** of posets, or preorders, the idea of an adjunction emerges in a particularly simple form. Let $\mathscr{P} = (P, \leq_P)$ be a preordered set, regarded as a category, i.e., with the ordering relation on the objects of P given by

$$x \leq_P y \text{ iff there exists an arrow } x \to y.$$

Let $\mathscr{Q} = (Q, \leq_Q)$ be another preorder, with an ordering relation defined similarly. We know that covariant functors between such categories are just monotone (order-preserving) functions, and that contravariant functors are antitone (order-reversing) functions. Suppose we have a pair of monotone maps $F : P \to Q$ and $G : Q \to P$,

$$P \underset{G}{\overset{F}{\rightleftarrows}} Q$$

such that for all $a \in P$ and $b \in Q$, we have the two way rule

$$\frac{F(a) \leq b}{a \leq G(b)}$$

where the bar indicates 'iff'.[43] If such a condition obtains, the pair $\langle F, G\rangle$ forms a (monotone) *Galois connection* between the preorders (posets) $\mathscr{P}$ and $\mathscr{Q}$. We also say that F is the *left (or lower) adjoint* and G the *right (or upper) adjoint* of the pair, and write $F \dashv G$ to indicate such an adjunction. Moreover, it follows immediately from the existence of such a Galois connection (simply use the identities $Fa \leq Fa$ and $Gb \leq Gb$, i.e., set $b = Fa$ and $a = Gb$ respectively) that $a \leq GFa$ and $FGb \leq b$. This basically describes the following notions: for each $p \in P$, we call the *unit* of the adjunction an element $p \leq GFp$ that is least among all x with $p \leq Gx$; dually, for each $q \in Q$, the *counit* is an element $FGq \leq q$ that is greatest among all y with $Fy \leq q$.

Example 6.0.14. If we consider $\mathbb{Z}$ and $\mathbb{R}$ as posets with the standard ordering $\leq$ and the poset inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$, then the latter inclusion can be seen to have for left adjoint the ceiling function and for right adjoint the floor function. In more detail, for $n \in \mathbb{Z}$ and real $r \in \mathbb{R}$, we have that $n \leq r$ iff $n \leq \lfloor r \rfloor$, with $\lfloor r \rfloor$ denoting the greatest integer less than or equal to r. Similarly, $r \leq n$ iff $\lceil r \rceil \leq n$. Thus, we have

$$\begin{tikzcd}
\mathbb{R} \arrow[r, bend left=40, "\lceil - \rceil"] \arrow[r, bend right=40, "\lfloor - \rfloor"'] & \mathbb{Z} \arrow[l, "i" description]
\end{tikzcd}$$

[43]For the contravariant (antitone) version, we would have $b \leq Fa$ iff $a \leq Gb$.

An adjunction is a straightforward generalization of the previous notion of a Galois connection.

Definition 6.0.13. An *adjunction* is a pair of functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ such that there is an isomorphism

$$Hom_{\mathbf{D}}(F(c), d) \cong Hom_{\mathbf{C}}(c, G(d)), \tag{6.6}$$

for all $c \in \mathbf{C}, d \in \mathbf{D}$, which is moreover natural in both variables. In this particular case, we say that F is left adjoint to G, or equivalently G is right adjoint to F, denoted $F \dashv G$; sometimes the morphisms $F(c) \xrightarrow{f^\sharp} d$ and $c \xrightarrow{f^\flat} G(d)$ of the bijection given above are called the *transposes* of each other. In saying that the isomorphism is "natural in both variables," we mean that for any morphisms with domain and codomain as below, the square on the left commutes (in $\mathbf{D}$) iff the square on the right commutes (in $\mathbf{C}$):

$$\begin{tikzcd} F(c) \arrow[r, "f^\sharp"] \arrow[d, "F(h)"'] & d \arrow[d, "k"] \\ F(c') \arrow[r, "g^\sharp"'] & d' \end{tikzcd} \quad \Longleftrightarrow \quad \begin{tikzcd} c \arrow[r, "f^\flat"] \arrow[d, "h"'] & G(d) \arrow[d, "G(k)"] \\ c' \arrow[r, "g^\flat"'] & G(d') \end{tikzcd}$$

Following the example of the Galois connection definition, we can also define the notions of the unit and counit of an adjunction.[44] I briefly indicate one other interesting example of an adjunction.

Example 6.0.15. One can define modal operators in terms of adjunctions. If we let $i : M \hookrightarrow P$ be an order-preserving and injective map between the posets M and P, and if

[44]See Riehl, *Category Theory in Context* (121) or MacLane, *Categories for the Working Mathematician* for details.

we assume that i has a left and right adjoint, i.e., $p \dashv i \dashv n$, then we can consider the composite endo-maps $i \circ p$ and $i \circ n$ on P. If we regard P as the set of propositions ordered by the relation of 'following', M as propositions that are closed under modal operators, and p and n as the optimal ways of converting a (possibly contingent) proposition into one that is modally closed, then $\diamond = i \circ p$ and $\Box = i \circ n$ can regarded as the 'possibility' and 'necessity' operators. The operators, defined as above in terms of the endo-maps on P, obey the following relations:

1. $\Box \leq id_P \leq \Diamond$.
2. $\Box\Box = \Box, \Diamond\Diamond = \Diamond$.
3. $\Diamond\Box = \Box, \Box\Diamond = \Diamond$.
4. $\Diamond \dashv \Box$.

A few final general facts about adjunctions worth mentioning are that left (right) adjoints are closed under composition, i.e., given the adjunctions

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{D} \arrow[l, "G"] \arrow[r, bend left, "F'"] \arrow[r, phantom, "\perp"] & \mathbf{E}, \arrow[l, "G'"] \end{tikzcd}$$

the composite $F' \circ F$ is left adjoint to the composite $G \circ G'$:

$$\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F' \circ F"] \arrow[r, phantom, "\perp"] & \mathbf{E}. \arrow[l, "G \circ G'"] \end{tikzcd}$$

Moreover, arbitrarily long strings of adjoints can be produced,[45] and one can also define the notion of a morphism of adjunctions.[46] A map may or may not have a left (or right) adjoint;

[45] See Chapter 6 of Simmons, *An Introduction to Category Theory* for a few simple examples of such strings, especially the simplicial category example.

[46] See 4.2 of Riehl, *Category Theory in Context* for details.

the map may have one without the other, neither, or both (where these may be the same or different). Adjoint functors interact in particularly interesting ways with the limit and colimit constructions: in particular, *right adjoints preserve limits* (RAPL) and *left adjoints preserve colimits* (LAPC).[47]

Geometric Morphism Definition

The definition of geometric morphisms between toposes uses the concept of adjunctions and is rooted in the example of sheaves on topological spaces together with the fact that a continuous function $f : X \to Y$ between topological spaces induces a pair of functors—the inverse image functor f^* and the direct image functor f_*—such that $f^* \dashv f_*$, i.e.,

$$\begin{tikzcd} \mathbf{Sh}(X) \arrow[r, "f_*"'] \arrow[r, phantom, bend left=20, "\perp"] & \mathbf{Sh}(Y). \arrow[l, bend right=50, "f^*"'] \end{tikzcd}$$

Consideration of the definition of the inverse image functor f^* reveals that f^* preserves finite limits, i.e., that it is left exact. This motivates the following definition:

Definition 6.0.14. A *geometric morphism* $f : \mathscr{F} \to \mathscr{E}$ between toposes consists of a pair of functors $f^* : \mathscr{E} \to \mathscr{F}$ and $f_* : \mathscr{F} \to \mathscr{E}$ such that $f^* \dashv f_*$ and also f^* is left exact. In this case, we call f_* the *direct image* part of f, and f^* the *inverse image* part of the geometric morphism.[48]

If we have two geometric morphisms $f, g : \mathscr{F} \to \mathscr{E}$, a natural transformation

[47] Note also that, in general, a functor is said to be *left exact* if it preserves *finite* limits, and *right exact* if it preserves finite colimits.

[48] Following Grothendieck, the asterisk notation is meant to suggest functors that exist for every f, something that can be compared with functors that exist only for special sorts of f (where this is usually designated with an exclamation point). The lower position is meant to indicate functors having the same direction as f, while the upper position denotes functors going in the opposite direction of f.

$$\begin{tikzcd} \mathscr{F} \arrow[r, bend left, "f"{name=U}] \arrow[r, bend right, "g"'{name=D}] & \mathscr{E} \arrow[from=U, to=D, phantom, "\Downarrow \eta"] \end{tikzcd}$$

is given by a natural transformation $f^* \Rightarrow g^*$ between the inverse image parts (or equivalently, by adjunction, we could define this in terms of a map $g_* \Rightarrow f_*$ between direct image parts). Toposes together with geometric morphisms and the natural transformations between them forms a 2-category, where the objects are toposes, the 1-cells are the geometric morphisms, and the 2-cells are natural transformations as specified above.

Cohesive Toposes: Extended Examples

Example 6.0.16. For certain toposes, such as $\mathscr{M}$ some topos with some degree of cohesion or activity (e.g., the category of topological spaces or presheaves on a category) and $\mathscr{K}$ a topos devoid of any internal cohesion and variation (e.g., the usual category of sets), we can form the functor $\mathscr{M} \xrightarrow{\Gamma} \mathscr{K}$ which acts to pick out the set of points of $\mathscr{M}$, sometimes called the *points* or *global section* functor.[49] This functor has a left adjoint Δ, called the *discrete* or *constant* functor. For concreteness, taking $\mathscr{M} = \mathbf{Grph}$ and $\mathscr{K} = \mathbf{Set}$, Δ will take a set to the graph with as many oriented loops (arrows with source and target the same) as there are elements in the set, while Γ will take a graph to the set having for elements the oriented loops of the graph.

In certain cases, this adjunction pair $\Delta \dashv \Gamma$ gives a geometric morphism. For instance for $\mathscr{M} = \mathbf{Set}^{\mathbf{C}^{op}}$ and $\mathscr{K} = \mathbf{Set}$, such an adjunction pair is a geometric morphism, and since

[49]The ideas discussed in this example mostly began with Lawvere, and are discussed in Lawvere, "Tools for the Advancement of Objective Logic, Closed Categories, and Toposes"; Lawvere, "Cohesive Toposes and Cantor's Lauter Einsen"; and Lawvere, "Unity and Identity of Opposites in Calculus and Physics."

Set acts as a terminal object in the category that has presheaves for objects and geometric morphisms between them for morphisms, it can be shown that this pair (Δ, Γ) is the only geometric morphism from $\mathbf{Set}^{\mathbf{C}^{op}}$ into **Set**. More generally, one can think of the result of applying the discrete functor Δ as producing a subcategory of discrete spaces or as yielding a space with no cohesion, meaning basically that there does not exist a map from a connected space to it that passes through two distinct points. This suggests that perhaps there exists a further functor from $\mathscr{K}$ back to $\mathscr{M}$ that—at the other extreme of the result of applying the discrete functor—would yield a space of total (or infinite or trivial) cohesion. In certain cases, there does indeed exist such a functor $B : \mathscr{K} \to \mathscr{M}$, called the *chaotic* or *codiscrete* functor. The chaotic space produced by applying such a functor can be regarded as so "extremely" cohesive that, in moving a point to any other point, one need not concern oneself with the constraints put on the category by how the motion is parameterized (or the cohesion determined). As Lawvere claims: "we may say that points in a discrete space are *distinct*, while points in a chaotic space are *indistinguishable* if chaotic spaces are connected."[50] For concreteness, when $B : \mathbf{Set} \to \mathbf{RGrph}$, i.e., lands in the category of reflexive graphs, $B(S)$ will yield a vertex for each of the elements of S and for each pair (x, y) of elements of the set S, the functor will yield a unique arrow with source x and target y. In other words, $B(S)$ just describes the *complete graph* on the elements of the set S.

A further feature of the geometric morphism given by the adjoint pair (Δ, Γ) is that, in certain cases, it is *essential*, meaning that there will exist a further functor Π, sometimes called the *connected components* functor, such that $\Pi \dashv \Delta \dashv \Gamma$. In particular, we have

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[50] Lawvere, "Cohesive Toposes and Cantor's Lauter Einsen," 9.

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{op}} \arrow[r, bend left=50, "\Pi"] \arrow[r, bend right=50, "\Gamma"'] & \mathbf{Set} \arrow[l, "\Delta" description] \arrow[l, phantom, shift right=3ex, "\perp"] \arrow[l, phantom, shift left=3ex, "\perp"]
\end{tikzcd}$$

This is actually simply a restatement of an earlier result, since here $\Pi = \text{colim}$ and $\Gamma = \lim$. The definition of Π is forced by the fact that it must preserve colimits (the glueings); and by considering particular categories, such as the category of graphs, it is easy to see how Π just picks out the connected components. In a more general context, this functor can also be thought of as assigning to each object in $\mathscr{M}$ the cardinal representing the number of maps it supports into discrete sets. In certain cases, i.e., for certain special toposes $\mathscr{M}$ and $\mathscr{K}$, we can construct the adjoint quadruple diagram:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, bend left=60, "\Pi"] \arrow[r, bend right=20, "\Gamma" description] & \mathscr{K} \arrow[l, bend right=20, "\Delta" description] \arrow[l, bend left=70, "B"]
\end{tikzcd}$$

In particular,

$$\begin{tikzcd}
\mathbf{Set}^{\mathbf{C}^{op}} \arrow[r, bend left=60, "\Pi"] \arrow[r, bend right=20, "\Gamma" description] & \mathbf{Set} \arrow[l, bend right=20, "\Delta" description] \arrow[l, bend left=70, "B"]
\end{tikzcd}$$

For an $\mathscr{M}$ where variation/dynamics is more relevant than cohesion, the same adjunctions are called orbits $\dashv$ stationary points $\dashv$ equilibria $\dashv$ chaotic.

Restricting our attention just to the adjoint functors $\Delta \dashv \Gamma \dashv B$, we can consider maps from the composite $\Delta\Gamma(M)$ to some M in $\mathscr{M}$ as well as maps from M to the composite $B\Gamma(M)$, i.e.,

$$\Delta\Gamma(M) \longrightarrow M \longrightarrow B\Gamma(M).$$

The space $\Delta\Gamma(M)$ on the left can be thought of as the closest approximation to M 'from the left' given only its cardinality (points), while the space $B\Gamma(M)$ can similarly be thought of as its approximation 'from the right'. For each object in the domain of the points functor, these maps provide an interval between which the object must lie, the endpoints being the two opposite subcategories, an interval that is in some sense relative to what the specific points functor does to the object on which it acts. Now, if we apply the points functor Γ again, we get a sequence of isomorphisms (in $\mathscr{K}$)

$$\Gamma\Delta\Gamma(M) \xrightarrow{\cong} \Gamma(M) \xrightarrow{\cong} \Gamma B\Gamma(M).$$

But this just says that even though the two composite maps are in general not isomorphisms in $\mathscr{M}$, applying the 'points' functor yields an isomorphism of cardinals (in $\mathscr{K}$). The cardinal $\Gamma(M)$ or points(M) associated to a given M is at once isomorphic to the cardinal associated to the space $\Delta\Gamma(M)$ and to the cardinal associated to the space $B\Gamma(M)$. However, in the case of $\Delta\Gamma(M)$, all the points will be distinct, while in $B\Gamma(M)$ all points will be indistinguishable. In other words, we may have a definite number of points, however via the unifying isomorphism such points will be indistinguishable by any property. Lawvere remarks how this precisely captures the apparent paradox (first isolated by Cantor) that in an abstract set, all elements are distinct yet indistinguishable.

The basic idea, then, is that $\mathscr{M}$ contains two opposed subcategories (the discrete and codiscrete objects), which are rendered identical through the category $\mathscr{K}$. In more detail, the composite of the counit and the unit precisely expresses the unity of opposites, i.e.,

$$\Gamma\Delta = \mathrm{Id}_K = \Gamma B.$$

Lawvere interprets this "productive inconsistency," namely of having a definite number of points without these points being distinguishable by any property, in terms of Hegelian dialectics. The basic idea is captured by the following diagram of natural transformations between the composite counit and unit functors:

$$\text{opposite}_1 \longrightarrow \text{unity} \longrightarrow \text{opposite}_2.$$

In the particular case of the topos of reflexive graphs, Lawvere observed that while the notions of discrete and codiscrete are dual there, the full subcategories of discrete graphs and codiscrete graphs are each equivalent to the category of sets; moreover, both the discrete and codiscrete graphs are identical when regarded in the category of all graphs, for which reason Lawvere spoke of such adjunction pairs as embodying Hegel's idea of the unity and identity of opposites. More generally, in the case of any such configuration, Lawvere speaks of *adjoint cylinders*, where the three functors involved are adjoint and the two composites are isomorphic to the identity in $\mathscr{K}$. Put otherwise: we have an adjoint triple where there are two parallel functors that are adjointly opposite in that they are full and faithful, and moreover there exists a third functor that is left adjoint to one of them and right adjoint to the other functor; as subcategories included in the ambient "containing" category, they are opposite, but by neglecting the inclusions they are identical. In more detail, in the particular case of graphs, the category of sets gets embedded in the category of graphs through the action of the functor Δ (producing discrete graphs) and the action of the functor B (producing codiscrete graphs). These notions are dual; however, the resulting full subcategories of discrete graphs and codiscrete graphs are further equivalent to the category of sets, thereby yielding the "identity." From the perspective of $\mathscr{M}$, the discrete and the codiscrete are united; looked at from the other end, $\mathscr{K}$ is identified with $\mathscr{M}$ via inclusion of subcategories in two opposite ways. One might also think of this in terms of Hegel's discussion of *quantity* as the dynamic unity of the "moments" of discreteness and continuity.[51]

To make these ideas a little more concrete, consider the following.[52] If we set both $\mathscr{M}$ and $\mathscr{K}$ as the poset of natural numbers $\mathbb{N}$ (viewed as a category),[53] we can construct the two parallel functors $E, O : \mathbb{N} \to \mathbb{N}$, defined by $E(n) := 2n, O(n) := 2n + 1$, i.e., 'even' and 'odd' functions. These functors obviously act to produce the two subcategories $\mathbb{N}_{even}$ and $\mathbb{N}_{odd}$ of $\mathbb{N}$, which is another way of saying that both functors are full and faithful. Such subcategories clearly are "opposed" to one another, at least in the sense that $\mathbb{N}_{even} \neq \mathbb{N}_{odd}$; however, performing the same sort of simple composite applications as above, we can produce the bijection $\mathbb{N}_{even} \xrightarrow{\cong} \mathbb{N}_{odd}$, through which they can be viewed as "identical." As subcategories, both can be seen to be "united" as the opposing parts of the containing category $\mathbb{N}$, in relation to which, by virtue of each being isomorphic to one another through their isomorphic maps to $\mathbb{N}$ itself, they are rendered identical. There indeed exists a third "middle" functor $T : \mathbb{N} \to \mathbb{N}$ that, together with E and O, will form the appropriate adjoint triple:

$$\begin{tikzcd} \mathbb{N} \arrow[r, "T" description] & \mathbb{N} \arrow[l, bend right, "E"', ""{name=A}] \arrow[l, bend left, "O", ""{name=B}] \end{tikzcd}$$

$$\begin{tikzcd}[row sep=0] & \\ \mathbb{N} \arrow[rr, "T" description] & & \mathbb{N} \arrow[ll, bend right=40, "E"'] \arrow[ll, bend left=40, "O"] \end{tikzcd}$$

By definition of adjunctions, and given that we are working with posets, $E \dashv T$ and $T \dashv O$ just means that $E(n) \leq m$ iff $n \leq T(m)$ and $T(n) \leq m$ iff $n \leq O(m)$. Moreover, as long as

[51]See Hegel, *Georg Wilhelm Friedrich Hegel*, II.1 ("Quantity").

[52]This very simple, but illustrative, example of the adjoint cylinder construction is inspired by Lawvere Message to catlist. My exposition also follows *nLab: adjoint modality* and *nLab: Aufhebung*.

[53]Technically such categories are not even toposes. The reader who cannot see why is invited to revisit the earlier section introducing toposes and try to see what makes them not qualify as toposes.

T exists, we will further have that $TE = \mathrm{Id} = TO$, which just means in our particular case that $T(2n) = n$ and $T(2n+1) = n$, which forces the following piecewise definition of T as

$$T(k) = \begin{cases} \frac{k}{2} & \text{if } k \in \mathbb{N}_{even} \\ \\ \frac{(k-1)}{2} & \text{if } k \in \mathbb{N}_{odd}. \end{cases}$$

The idea here is that the adjoint triple $E \dashv T \dashv O$ all at once embraces the *identity* by the natural isomorphism $TE \xrightarrow{\cong} TO$, the *opposition* by the induced adjunction $ET \dashv OT$, and the *unity* by the idempotent relations $(E \circ T) \circ (E \circ T) = E \circ T$ and $(R \circ T) \circ (R \circ T) = R \circ T$. In this way, the middle functor T can be thought of as simultaneously identifying, opposing, and uniting E and O.

In the case of the category of presheaves on $\mathbf{C}$, Δ yields the discrete presheaves on $\mathbf{C}$ while B yields the codiscrete presheaves on $\mathbf{C}$, and Γ is their common projection. With such a set-up, it is always the case that $\Gamma\Delta \cong Id \cong \Gamma B$. In the particular context of presheaves, Lawvere takes the Hegelian notion of Aufhebung still further to yield a theory of dimension or "levels." Roughly, a level is a functor from a given category into one that is "smaller," that moreover has both left and right adjoint sections which produce subcategories that in themselves are identical (in the smaller category) but which include themselves as subcategories in opposite ways (and which, moreover, give rise to the two composite idempotent functors on the given, "larger," category). More specifically, given an adjoint cylinder situation between toposes, a *level* of a topos is defined as the inclusion of the right adjoint in this set-up. In Lawvere's approach, the *Aufhebung* of a level will be the smallest level that acts to resolve the component opposites (the opposing functors).[54] It is not the case that such an 'Aufhebung-like' level always exists for any given level, however in particular cases such as presheaf toposes over graphic categories, it does exist.[55]

Given more space, this discussion would lead very naturally to a more thorough exposition of the notion of a *cohesive topos*, in which context further adjoint cylinders (of special significance to the characterization of the concept of continuity) arise, including situations relating to infinitesimally-generated spaces; however, we leave the curious reader to pursue these more advanced matters on their own.[56] For now, we content ourselves with observing that one sometimes finds further "adjoint cylinders" embedded "in between" $\mathscr{M}$ and $\mathscr{K}$ via intermediate categories $\mathscr{L}$ that are less "abstract" than **Set** but with a simpler sort of cohesion than $\mathscr{M}$:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, shift right=1.5, "\Gamma"'] & \mathscr{L} \arrow[l, shift right=1.5, "\Delta"'] \arrow[l, bend left=60, "B"] \arrow[r, shift right=1.5, "\Gamma"'] & \mathscr{K} \arrow[l, shift right=1.5, "\Delta"'] \arrow[l, bend left=60, "B"] \arrow[ll, bend right=50, "\Delta"'] \arrow[ll, bend left=70, "B"]
\end{tikzcd}$$

The basic idea here is that various toposes such as $\mathscr{M}$ or $\mathscr{L}$ are contrasted with the extreme case of $\mathscr{K}$ via geometric morphisms; but the diagram above suggests that we extend this to consider intermediate toposes and chains of maps between such adjoint triples, with the

[54]For more details on this theory of levels, see Lawvere, "Display of Graphics and their Applications Exemplified by 2 Categories and the Hegelian Taco," Lawvere, "Unity and Identity of Opposites in Calculus and Physics," and Kennett et al., "Levels in the toposes of simplicial sets and cubical sets." In the toposes of simplicial sets, cubical sets, and reflexive globular sets—each definable in terms of presheaf categories—levels coincide with the notion of dimension.

[55]See Lawvere, "Linearization of Graphic Toposes via Coxeter Groups" for details. "Graphic categories" can just be thought of as certain "simple" enough categories that allow for finite graphic display or presentation once one constructs their corresponding presheaf category.

[56]See Lawvere, "Axiomatic Cohesion."

effect that the various determinations of cohesivity or variation in toposes themselves can be compared.

Philosophical Pass (7th): Cohesiveness; A New Dialectical Science?

I now summarize what I take to be the most significant results to have come out of the preceding discussion. An object that arises in a "spatial" category (a category with some cohesion or variation) can be examined via *levels*, constructions that provide a precise formulation of the unity and identity of opposites so characteristic of the (originally vaguely formulated) philosophical concept of dialectics, making high-level relations between generalities and entire contexts (or "universes") amenable to exact solution. In the above exposition, we even saw how, from one perspective, the moments of discreteness (zero cohesion) and continuity (total cohesion) could be unified. However, this was a rather extreme case. By considering intermediate cylinders and passages between adjoint triples (or quadruples) of various toposes, we move beyond the case of relating a single category to the extreme case of discreteness/constancy (as in $\mathscr{K}$). We can now examine sequences of intermediate categories that are interlocked via cylinder maps of their own, opening onto a more refined dialectical science of cohesion, by which ultimately one could systematically characterize and compare the differing properties of cohesion and variation that emerge in certain "universes" or models for mathematical theories that treat of objects with some dynamics. Using such intermediate adjoint cylinders, the quality of dimension or level in spaces can be compared.[57]

[57]See Lawvere, "Tools for the Advancement of Objective Logic, Closed Categories, and Toposes" for more details and some examples of this.

At a fundamental level, what is going on here is that we are claiming that the question of what is more variable, more cohesive—or how various models for certain mathematical theories dealing with dynamical phenomena are differently variable or cohesive—should involve functorial comparisons. The further suggestion can be made that, in addition to examining such sequences of adjoint cylinders, through the discovery and study of *left exact* functors between two toposes (which may not have adjoints) we should be able to make even more precise the notion of greater or lesser discreteness (non-cohesion or constancy) vs. continuity (cohesion or variability). In a sense, this latter suggestion is a natural extension of the perspective of the morphological notion of continuity via (co)continuous functors (defined as preserving limits, and not just those that are finite). Going somewhat further, we could perhaps use such functors to begin to construct a metric measuring something like 'what it would cost' to make an 'almost-adjunction' an honest adjunction; another use of the resulting metric might be to begin to more precisely analyze 'how far' we are from the more extreme or trivial cases of "infinite" cohesion (continuity) on the one hand and zero cohesion (discreteness) on the other. More generally, this sort of approach should open onto a much richer terrain of dialectical subtleties, and has the potential to provide a powerful framework for the measurement of the continuity-discreteness of an entire setting or context in which objects are studied.

Finally, if the ancient concept of dialectics is indeed more properly developed in terms of adjoint triples (quadruples), the 'higher-order' sort of comparisons between such adjoint triples (or quadruples) might be thought of as not just yielding a manner of studying the relative properties of cohesion/variation of differing toposes, but—following Lawvere's use of Hegel's notion of *abstract generals* and *concrete generals*[58]—we could say that via such interlocking sequences of maps between cylinders, the manner in which different concrete generals represent their objects as conforming to a shared *state of becoming* (such as involving internal dynamics, memory, some topological features) can be compared.[59]

If we were forced to decide on such matters, then, such an emergent "science of cohesion" suggests that the question of whether "the universe" is fundamentally discrete or continuous is not exactly well-posed; rather, the continuity (discreteness) of a particular setting is properly captured by the types of passages admitted between concrete "universes" (specific categories or toposes) that, relative to one another, support differing degrees and properties of cohesion (non-cohesion) and variability (constancy).

Conclusion

I conclude this chapter with a table summarizing the philosophical or conceptual contributions of this chapter (in terms of continuity-generality).[60]

[58]The latter being categories thought of as a sort of presentation of all the possible objects of some abstract type together with all the transformations and comparisons supported between these possible objects.

[59]I am here grossly oversimplifying Lawvere's appropriation of the Hegelian notions of abstract generals, concrete generals, etc. The reader who desires a more thorough account should consult Lawvere, "Tools for the Advancement of Objective Logic, Closed Categories, and Toposes" and *Sheaf Theory Through Examples*, where I develop more fully the connections between the adjoint dialectics initiated by Lawvere and this use of the philosophical concepts of (abstract or concrete) generals.

[60]In addition to each of the 7 themes discussed separately in the 7 "philosophical passes," the table includes three extra perspectives; while some of these were briefly alluded to, full discussion of these three other perspectives was omitted from these chapters in the interest of space. The curious reader can consult *Sheaf Theory Through Examples* for more details on the remaining 3 (as well as a few other perspectives).

Perspectives	***Continuity***	***Generality***	***Consult Section***
(1) Yoneda and Representables	Continuity as **Relationality**	Objects represented generically	1st Philosophical Pass (Chapter 6)
(2) Every Presheaf a Colimit of Representables	Continuity as **Mediation**	Colimit as a universal "gateway" property	2nd Philosophical Pass (Chapter 6)
(3) Action of the Presheaf	Continuity via: (1) **dynamism/process**; (2) **boundary-extraction**; (3) **restriction**	(1) objects evolve over stages; (2) isolation of generic figures; (3) regularity of parts-whole relation	3rd Philosophical Pass (Chapter 6)
(4) Sheaves (Restriction-Collation)	Continuity as **Local-global passage**	Local-global passages transforms "How to get from the many (particulars) to the one (general/universal)?"	4th Philosophical Pass (Chapter 6)
(5) Sheaf Cohomology	Continuity as **absence of obstructions to local-global passages**; Discreteness as existence of obstructions	Vanishing cohomology groups as globalizability of local systems; Non-vanishing groups as indicating their particularity/contextuality	5th Philosophical Pass (Chapter 6)
(6) Limit Preservation	Continuity as **morphological** (preservation of privileged objects acting as intermediaries in their respective category)	(Co)Continuous functors as preserving certain universal objects	Definition 6.0.7 and following (Chapter 6)
(7) Pointless Topology; Locales; Intuitionistic Logic	Continuity **without points**	Sheaves on a locale gives rise to generalized set-theory; logic of sheaves in general is intuitionistic	6th Philosophical Pass (Chapter 6)
(8) Grothendieck Toposes; Sheaves over a Site	**Continuous transformations extended** beyond topological contexts; G. toposes as **unifying the discrete and the continuous**	Grothendieck topos as a generalized topological space; geometric morphisms as generalized continuous maps	6th Philosophical Pass (Chapter 6)
(9) New Perspectives on the Continuum	In a certain topos, **all real-valued functions are continuous**; in a topos model for SIA, **the real-line continuum R is non-detachable**	In the same topos model for SIA that supports infinitesimal objects, the law of the excluded middle fails in general (but holds for closed formulas)	
(10) Cohesive Toposes	Continuity as (type of) **Cohesion/Variability**	Adjoint cylinders and passages between them formalizing the notion of dialectics; might also interpret such maps as informative passages between distinct "concrete generals"	7th Philosophical Pass (Chapter 6)

Chapter 7

Conclusion

Let us return to the question that began this dissertation: is the universe continuous or discrete? Anyone attempting to answer this question must acknowledge the many examples of systems comprised of discrete components that give rise to continuous behavior as well as continuous systems giving rise to discrete behavior or changes. There are many examples of continuity emerging out of the discrete. Fluid flow appears continuous (and is typically held to be described by continuous equations), yet real fluids consist, at an underlying level, of discrete particles in random motion.[1] Discrete grains of sand and other granular materials flow in a way comparable to how continuous fluids flow, and collectively exhibit many behaviors similar to continuous fluid flow, as with sand dunes compared to ocean waves. A movie or motion picture is a finite number of distinct still images, presented one after the other, yet when a certain threshold speed is crossed, the appearance of a continuous moving image is produced. Our eyes scan the horizon *cut by cut*, taking in discrete sensory inputs, and our perception system synthesizes these discrete inputs, interpolating from such inputs and filling in missing information.[2] Similarly, when viewing a photograph or television,

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[1]Yet computation of fluid flow, using continuous models, is done by approximating with a discretized PDE or some form of discrete mesh, so computation of fluid flow is also arguably discrete, despite what the continuous form of the equations suggest.

[2]Researchers refer to this as "perception completion." This process, sometimes called the "continuity illusion" in the literature, is not restricted to our visual systems but also occurs with other sensory modalities, for instance:

> when talking with a friend, a sudden cough from someone behind you may mask the friend's speech sound, but you can still hear and understand what he says. Even when a portion of the foreground sound is completely removed and replaced with a loud noise, listeners believe they hear the sound continuously behind the

one can have the impression that everything one sees in the photograph or on the screen is continuous, yet really there is a point lattice where individual cells are being selectively illuminated by something like a "spotlight" scanning across the grid. Examples of this sort might suggest the idea that, indeed, all of nature is something like a television screen: whenever one observes continuity, there is really only discrete particles.

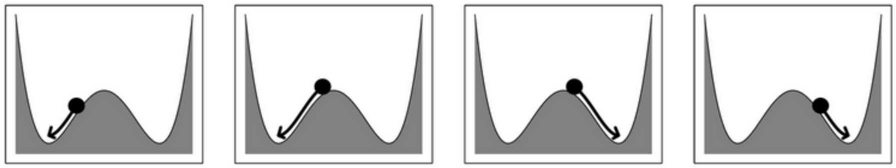
On the other hand, there are many examples of discreteness emerging from continuity. A very conspicuous example of this occurs in phase changes, where, for instance, water changes temperature continuously until, suddenly, this gradual succession is interrupted by a change in state (such as when boiling water turns from a liquid into a gas, or when cooling water suddenly freezes and solidifies).[3] On the basis of such changes in state, one might be inclined to conclude that, at least some of the time, nature does "make leaps." We stretch a rubber band, deforming it continuously, and then we stretch it a little more, still continuously, and suddenly it snaps and breaks. Moreover, one can have a continuous system, such as that modeled by a continuous curve or "hump" over which a ball is rolled continuously, in which discrete outcomes or behaviors are observed as one varies the initial conditions in a continuous fashion.[4]

> interrupting sound, which is the so-called auditory continuity illusion. [...] The phenomenon [of auditory continuity] has been reported in both humans... and also in monkeys[...]. (Kitagawa, Igarashi, and Kashino, "The tactile continuity illusion," 1784)

These authors mention how recent studies have reported a similar process of "perceptual completion" and the "continuity illusion" across other sensory modalities, for instance in hearing and touch. See ibid. for details.

[3]Of course, these transitions do not happen *everywhere* at once.

[4]This example and the following image comes from Wolfram, *A New Kind of Science*, 341.



If the ball starts anywhere to the left of the center, it rolls to the minimum on the left; if it starts anywhere to the right, it rolls to the minimum on the right side. The point is: this is a continuous system in which a continuous change in initial conditions (position of the ball) produces a discrete change in behavior (rolling to the left or right).

There are also partial differential equations continuous in space and time, such as that given by the Schrödinger Equation, that yield discrete items like electron orbitals. On the other hand, there are a number of fundamentally discrete models, such as given by cellular automata, that evolve to emulate continuous behavior. Furthermore, there are frequently both viable continuous descriptions and discrete descriptions or models of the same phenomenon.[5] Other times, there are single models or approaches that involve a back-and-forth between the continuous and the discrete, as in the modeling of a problem with fuzzy control systems.[6] On the other hand, there can be a number of difficulties in passing from the continuous to the discrete (and vice versa), for instance as one observes with the general difficulty of getting AI systems to translate visual input into lists of objects, or

[5] To give just one example: models of microscopic fracture or breaking of materials. One model looks at arrays of atoms, and another is based on continuum descriptions of materials.

[6] In constructing such systems to deal with various applications, there is (i) a partition of the interval spanned by each identified variable into a number of fuzzy (continuous) subsets; (ii) an assignment of a membership function for each fuzzy subset; (iii) an assignment of fuzzy relationships between the inputs' fuzzy subsets and the outputs' fuzzy subsets, forming a "rule-base"; (iv) a "fuzzification" of the inputs to the controller; (v) a use of "fuzzy approximate reasoning" to infer the output contributed by each given rule; (vi) an aggregation of the fuzzy outputs supplied by each rule; and (vii) a process of "defuzzification" to form a crisp (discrete) output.

with the incapacity of classical (discrete) logic to deal with certain phenomena involving continuous variation. And there are even various curious "mixings" of the continuous and discrete to be found in the historical register, for instance as seen in the somewhat ironic fact that the word "calculus," which is now basically the "poster-child" of continuous systems, is derived from the word *calx*, a small pebble or stone used for counting and doing discrete calculations.

In short: it would seem to be incredibly naive to expect the entire universe, in every way, to be continuous *or* discrete. It appears to be a fact that there are continuous phenomena that evolve from discrete components, just as there are discrete phenomena or behaviors that evolve from continuous systems. One thing, at least, is clear from this: in asking if the universe is continuous or discrete, the question cannot be understood to refer to *all* things, at all "levels." Rather, the question is clearly aimed at a specific scale or level—usually at one held to be somehow more "fundamental" than the rest. In other words, there is an underlying assumption that there is some level or scale that is *special* in the sense that its continuity or discreteness would somehow be decisive in determining the continuity or discreteness of the universe as a whole—even while allowing that there are both continuous and discrete phenomena to be found throughout nature, that discrete systems can give rise to continuous behavior (and vice versa). In other words, once "unpacked," the question demands an answer of the following form:

- There is an object (broadly construed) A, occupying a specific "level," the continuity-discreteness of which would be of the highest relevance in determining the continuity-discreteness of the universe or nature.

- Ultimately, all A must have a continuous (or discrete) form of representation, since A is itself continuous (or discrete).
- All change in A is the result of continuous (or discrete) A-type processes.

In referring to a level or scale, I do not mean to implicitly restrict attention to some material or particle of a certain size. I might equally have spoken of a definite "plane of reality." By this I mean to suggest that, in answering the question, what is substituted in for A need not be anything like an *atom* or particle-like entity of a certain size, but is meant also to include items like, for instance, some basic unit of *information*, some basic ingredient of "computation," or some fluid-like "field" object. We are used to thinking that the most meaningful answer to the main question would come at the "level" of small scales, e.g., at the Planck length or below (around 10^{-35}m). But, logically at least, I see no reason why this need be the case. There need only be a level or "basic unit" (whatever its size or even if this is not a size at all, but is determined analytically) that is held to be special with regard to other levels or scales.

In general, then, one could answer (or dispute an answer) along the following lines:

- There is no specific "level" the continuity-discreteness of which could be relevant to determining the continuity-discreteness of the universe or nature. Therefore, whatever is substituted for A will be deficient.
- It is meaningful to hold that there is a specific "level" the continuity-discreteness of which would be of the greatest importance in determining the continuity-discreteness of the universe, but A is not the right sort of thing to look at, i.e., is not of the highest relevance.

- A is the right sort of thing to look at, but it is not to be given a continuous (or discrete) representation, but rather a discrete (or continuous) representation, since A is itself discrete (or continuous).
- A is the right sort of thing to look at, but it need not be given an exclusively continuous (or discrete) representation, but should rather be represented as both continuous and discrete, reflecting the fact that A itself exhibits both continuity and discreteness.
- A is the right sort of thing to look at, and it must have a continuous (or discrete) form of representation, yet A itself is discrete (or continuous or both), i.e., the best form of the representation need not follow the nature of the underlying phenomenon.
- A is the right sort of thing to look at, and it must have a both continuous and discrete form of representation, yet A itself is discrete (or continuous), i.e., the best form of the representation need not follow the nature of the underlying phenomenon.[7]

Then, assuming any of the above responses to the first prong of the answer (answers that take A to be the right sort of thing to be looking at, or substituting some B that is held to be the right sort of thing to be looking at), one could respond to the second prong as follows:

[7]The positions against which this, and the previous, claim are directed will have to supply an argument and find evidence for the assumption that the optimal form of representation must, with respect to continuity-discreteness, take on the same features characteristic of the underlying phenomena.

- A-type processes are the wrong processes to be looking at, and they cannot adequately explain change in A.
- A-type processes are the right processes to be looking at, and all change in A is the result of A-type processes, but these processes are discrete (or continuous), in contrast to what the given answer stipulates.
- Not all change in A is the result of continuous (or discrete) A-type processes, but some other, additional processes are operative, i.e., changes in A need not be the result of A-type (continuous or discrete) processes alone.

Regardless of what level is selected, populated by whatever A is selected, the important question will arise as to why or how it can be the case that, if at, e.g., very small scales, there is discreteness, this discrete "micro-structure" can not only give rise to phenomena that appear continuous but how this structure does not seem to have left a clear mark on the observable world on all larger scales, or how there can be continuous symmetries at various scales. In other words, in selecting such a level and designating it as "special," one must already have an idea about how the continuity or discreteness of the remaining "levels" are affected (or not) by the characteristic continuity or discreteness of this level. So even though the basic idea is frequently just that, in designating the special level as, for instance, continuous, the moments or other levels of discreteness found throughout nature are held to be "derived" or emergent, one must still be able to provide an account of how this "derivation" or emergence of the discrete in relation to the continuous (or vice versa) works.

One might be tempted by one "short-cut" approach to the apparent fact that continuity can give rise to discreteness and discreteness to continuity. We could assume that the continuous-discrete polarity reflects two ineliminable human needs: the need to *see* or *perceive* (continuous); and the need to *understand* (which is a finite, discrete process). Further assuming a kind of Parmenedian position—that 'being' and the *thought* thereof are one—this might further suggest an answer to the question: that everything is in fact discrete, and operates "digitally," including our brains, and all perceptions or measurements are discrete items; so a continuously evolving universe is an illusion produced when, say, the speed of our successive (fundamentally discrete) measurements or thoughts exceeds a certain threshold. Perhaps one could then even supply some naturalistic explanation (or even competitive advantage) behind the production of such an "illusion." But notice that even in describing continuity in this way—as some sort of "illusion" produced once a certain speed threshold is exceeded—there is a reliance on a very specific characterization of continuity (and presumably something that only arises in the particular relation between perceptibles and beings capable of perception).

Even leaving aside the difficult questions related to the existence of a "special level" and its relation to other "levels," when we dispute the continuity or discreteness of A and A-type processes, I think it should be a minimal requirement that there be some clarity about just what we are attributing to such things and whether that is the best characterization (and in what sense, according to what desiderata, it is "best"). In other words, for the question, possible answers, and disputes between proponents of different answers to be meaningful, we should hope to have some definite, sufficiently sophisticated and nuanced, but also ultimately widely agreed-upon sense of continuity and discreteness. Yet there are many models of continuity and discreteness, and it is not even immediately clear what standards we are to use in discriminating between "better" characterizations of continuity-discreteness.

On the other hand, most people do seem to have a decent working understanding and rough consensus regarding these terms, suggesting there is some collection of overlapping "intuitions" deployed by most people working on this question as they develop their respective characterizations.

Before hoping that an answer to the main question will be forthcoming, then, we should be sure to appreciate the various conceptualizations or models of continuity-discreteness, something that will allow us to begin to acquire a clearer view of the sorts of things we expect from such characterizations. In what remains of this dissertation, I thus describe a number of distinct models or characterizations of continuity.[8] I regard nearly all of the individual characterizations to follow as distinct, even while there is (frequently) some overlap between certain characterizations; they are held to be distinct because there is at least one aspect of the characterization, or even just a particular choice of emphasis, that renders it distinct from the other characterizations. In general, for each item, I will do the following three things: (1) describe the characterization; (2) indicate a few reasons for endorsing it or some motivating examples; and (3) indicate some possible problems or weaknesses with it or conflicts with other characterizations. I then briefly discuss, for each of the five main groupings of characterization, what such a characterization means for the understanding of generality. These models or characterizations, organized into 5 main categories, are as follows:

1. Question of Scale
    a) Randomness

[8] I speak of characterizations of *continuity*, but I hope it is clear that the notions of continuity and discreteness are nearly always correlative. In nearly all cases, an adequate grasp of the characterization of continuity will suffice to immediately provide a characterization of what such an approach regards as discrete.

   b) Idealization

2. Relation of Parts

   a) Density

   b) Compositional

      (1) Self-Similarity

      (2) Arithmetized/Atomized

      (3) Connectedness

         (a) Indecomposability

   c) Reflexivity

   d) Regularity

      (1) Homogeneity

   e) Contiguity/Positional

   f) Issue of Distinction

      (1) Indistinctness/Vagueness

      (2) Fuzziness

      (3) Possibility

   g) Structural

      (1) Functorial Order-Preserving

      (2) Agreement/Concord

      (3) Cohesion

3. Closeness

4. Issue of Size

5. Passage

a) Limits

b) Local-Global Passage

Various approaches to the main question seem to fall back on some claim to the effect that continuity (or discreteness) is "better" than the alternative, even in the course of developing the characterization itself. Thus, after discussing each of these characterizations one by one, I will conclude with an outline of some of the main positions regarding reasons for holding continuity (or discreteness) to be "better":

1. Continuity is "Better"

    a) Pragmatic

    b) Epistemic

    c) Aesthetic

2. Discreteness is "Better"

    a) Deflationists

    b) Digitalists

    c) Computation Power

3. Neither is "Better"

    a) Antinomial

4. Both "Better"

    a) Dialectical

I now begin with a discussion of each of the individual characterizations (19 in total, falling under 5 main headings).

Question of Scale

The concept of scale is actually somewhat subtle. Roughly, we think of scale as measuring the extent or size or dimension of something (spatial, temporal, analytical, quantitative), such as a length, area, duration, or amount. More generally, but also in the study of complex systems, the scale concept can involve the notion of *levels* and can be used to refer to some combination of (i) a mereological level, where a whole or some component part of the whole is analyzed, and (ii) a level of observation (for instance, observing a system as a part or participant in that system, or externally). In this way, the concept of scale is also frequently closely linked with hierarchies, and with an ability to generalize or extend observations made at one level—regarding the relationships between the objects and processes that "inhabit" that level—to another level of the scale.[9] Scales have extent and also resolution, where extent refers to the magnitude of a dimension (spatial, temporal, etc.) deployed in measuring something and determining the outer boundaries of the measured phenomenon, and where resolution refers to the granularity or precision deployed in measurement. From the perspective of organisms in general, resolution can be defined as the finest component of an environment or system that can be differentiated by the organism, while extent represents the maximal range of perception where the relevant object can still be distinguished by the organism. For human beings, resolution is usually the finest unit of measurement, while extent represents the total area under consideration or the range of the measured phenomenon.

[9]Hierarchies can come in various types, e.g., inclusive hierarchies (groups of objects ranked as lower are contained in, or are subdivisions of, groups ranked higher), exclusive (groups of objects ranked lower are not contained in, or subdivisions of, those ranked higher), and constitutive hierarchies (groups of objects combined into new units that can then be combined into further units with their own properties). See Gibson, Ostrom, and Ahn, "The concept of scale and the human dimensions of global change" for more on the concept of scale in relation to hierarchies.

Consider how a soccer field has an area of roughly $5,000$ square meters—this is the area that concerns the soccer player or the landscaper or a property owner or the land surveyor. But if you are a bug crawling around, up and down, along the blades of grass, the more relevant area would be the total surface area of all the grade blades—this is far larger than the "soccer field area," by at least a factor of 100. If you are interested, moreover, in the sun's photons being absorbed by the chlorophyll in the grass, this larger surface area would also be the relevant area.[10] So, the soccer field is characterized by at least two area scales, that moreover differ by a large factor. Obviously, there are many other situations or phenomena that will lead to many more scales, such as the various scales (with different resolutions) that can be used to determine the length of a border between two countries. Additionally, in many actual applications and studies, one must resort to a two-(or multi-) scale model, for frequently no single mechanism seems capable of explaining patterns on all scales. On the other hand, there are certain phenomena in physics that can be said to be "true on all scales" or "scale-less."

Yet even when there are many scales that can be brought to bear on some phenomenon, it sometimes happens that there are certain characteristic spatial and temporal scales at which dominant patterns tend to emerge, leading some ecologists, for instance, to speak of the "characteristic scale" of certain ecological phenomena.[11] Moreover, more generally, there is nearly always some sort of upper or lower limit forming a range of validity for the measured object. And such "end effects" make it so that, even when dealing with scale-

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[10] This example comes from Schroeder, *Fractals, Chaos, Power Laws*, 61-62.

[11] See, for instance, Delcourt et al., 1983, Urban et al. 1987.

invariant phenomena, real-world data are never *exactly* scale-invariant. However, as some (such as Mandelbrot) have noted, many things, such as mountainscapes, are only interesting once they exhibit features such as cliffs, peaks, and valleys, on many length scales.[12]

Much more could be said about the notion of scale (and would need to be said, if one were to fully develop such an approach to continuity-discreteness). But, for our purposes, it is more to the point to note how this model effectively commits to the idea that the differentiation of continuous and discrete phenomena is, ultimately, a matter of scale, or is *scale-dependent*. In addition to the sorts of examples discussed above, this idea is perhaps partly motivated by more phenomenological "facts" such as how our overriding impression of many things is as continuous, while in many cases this is the case only if we refrain from looking very closely, or if we are content to take for granted our current mental models as the "normal" time and space scales (large spatial scales, short time scales). Spatially, we generally tend to smooth things out at larger scales; temporally, abrupt changes and poignantly discrete details tend to be smoothed out over large time scales. If you can imagine hiking up a cliff and pausing to look out at the sea, consider that your large-scale, "smoothed out" view misses all of the sharply defined and more minute structure of the surface of the sea; and the five minutes we spend looking over it is massively smaller than the time scale of the ocean system, so that while we may see little to no directed change in it, on the ocean's time scale "abrupt" events and radical system changes may be underway. Our experience of many phenomena also instructs us to believe that *one and the same* entity can be both discrete and continuous (given scale-dependence).

[12]The mathematical concept of self-similarity, as invariance against changes in scale or size, is relevant in this context.

The two models considered under this heading—Randomness and Idealization—are arguably two of the more powerful ways of spelling out this notion of the essential scale-dependence of continuity and discreteness.

Randomness

Without the presence of randomness, and when the individual components of a system are discrete, the structures or processes one observes at larger scales typically "mirror" or "reduplicate" the underlying discreteness that characterizes the system at microscopic levels. On the other hand, again even when the individual components of a system are discrete, if randomness is present and you look at the "average behavior" of the system at larger scales or at the level of a large number of interacting components, it may appear continuous and "smooth." Examples in natural systems abound, including how fluids such as water can behave continuously, yet at the small scale they are made up of discrete molecules in random motion. Such phenomena suggest the following idea: what enables systems that are made up of discrete components to behave in a continuous way is *randomness*. Stephen Wolfram, for instance, argues that

> If there is no randomness, then the overall forms that one sees tend to reflect the discreteness of the underlying components. Thus, for example, the faceted shape of a crystal reflects the regular microscopic arrangement of discrete atoms in the crystal.
>
> But when randomness is present, such microscopic details often get averaged out, so that in the end no trace of discreteness is left, and the results appear to be smooth and continuous. [...] [T]he randomness has in a sense successfully washed out essentially all the microscopic details of the system.[13]

[13] Wolfram, *A New Kind of Science*, 327-28.

Perhaps the most curious observation made by Wolfram, though, comes when he notes that it turns out that it does not really matter how randomness comes to be present in the system, whether it is "inserted" from the outside at each step, or arises intrinsically—all that really matter is that randomness is present. Fluid turbulence gives a nice example of the fact that this randomness is not a sensitivity to, or dependence on, initial conditions (as randomness is ordinarily defined).[14] While examples where this sort of thing occurs might lead one to wonder whether any system that involved randomness would produce, at a more "macroscopic scale," continuous patterns of growth, this does not in fact seem to be the case. Rather, it would be more accurate to say that

> it seems that continuous patterns of growth are possible only when the rate at which small-scale random changes occur is substantially greater than the overall rate of growth. For in a sense it is only then that there is enough time for randomness to average out the effects of the underlying discrete structure.
>
> And indeed this same issue also exists for processes other than growth. In general the point is that continuous behavior can arise in systems with discrete components only when there are features that evolve slowly relative to the rate of small-scale random changes.[15]

One could also approach this in terms of the presence of conserved quantities that force certain overall features not to change too quickly.[16] With the presence of both of these features—the rate at which small-scale random changes occur being substantially greater

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[14]See Wolfram, *A New Kind of Science*, 376. According to Wolfram, the fact that this "cannot simply be a reflection of randomness that is inserted through the details of initial conditions" can be interpreted to mean that "most of the randomness we see is not in any way inserted from outside but is instead intrinsically generated inside the system itself" (381).

[15]Ibid., 333.

[16]Conserved quantities refers to conservation laws, e.g., the fact that in the evolution of any closed physical system, the total values of quantities like electric charge or energy seem to stay the same.

than the overall rate of growth plus the presence of conserved quantities forcing some overall features not to change too quickly—continuous behavior can be produced from systems consisting of discrete components.

The overall idea, in sum, is that continuity can be generated from entities or systems with discrete components *whenever global features of the system evolve at a slower rate than small-scale random fluctuations* (something that may be helped by the existence of conserved quantities forcing overall features not to change too rapidly). In such cases, randomness acts to "average out" any discrete distribution characterizing the initial system at the underlying level, making the evolution of its behavior at larger scales look "smooth." Continuity is accordingly the result of a kind of "order-scrambling" at the ground level that works more quickly than higher-level evolutions or growth of stabilizing patterns.

These sorts of observations can be contrasted with an aspect of the local-global passage model (see below). In one way of developing the latter, the basic or fundamental processes are local: each "particle" or "cell" behaves in a way determined entirely by the state of its local neighborhood of "cells." Yet since randomness "washes out" discrete traits or aspects of the system observable at small-scales, so that "globally" there is no sign of the orientation and arrangement of the array at local levels, this Randomness characterization might be conceived of as the opposite of local-global passage: here, the feature of discreteness that characterizes things locally does *not* get "lifted" to more global levels—and that is precisely what characterizes its continuity.

A potential problem with this Randomness perspective is that, in making continuity a matter of growth rates of global features as compared to local "small-scale" random fluctuations of discrete components, it would appear to simply *assume* that, at a "fundamental" level or "small-scale," things are discrete. It does, however, have the virtue of providing a strong account, on the assumption that things are fundamentally discrete, of the emergence of continuous behavior.

Idealization

The basic idea here is that all phenomena that appear continuous are fundamentally discrete, but continuous models represent something like a useful "idealization," helping to "simplify" otherwise more computationally-demanding (discrete) approaches to more complex phenomena. Grothendieck describes such a view when commenting on Riemann:

> It could well be that the ultimate structure of space is discrete, while the continuous representations that we make of it constitute perhaps a simplification (perhaps excessive, in the long run...) of a more complex reality; that for the human mind, "the continuous" was easier to grasp than the "discontinuous", and that it serves us, therefore as an "approximation" to the apprehension of the discontinuous. This is a remark of a surprising penetration in the mouth of a mathematician [namely, Riemann], at a time when the Euclidean model of physical space had never yet been questioned; in the strictly logical sense, it was rather the discontinuous which traditionally served as a mode of technical approach to the continuous.

Leibniz also developed a similar view of the "ideal" nature of continuity, and accordingly insisted on distinguishing the "actual" and the "ideal":

> Matter is not continuous but discrete, and actually infinitely divided, though no assignable part of space is without matter. But space, like time, is something not substantial, but ideal, and consists in possibilities, or in an order of coexistents that is in some way possible. And thus there are no divisions in it but such as are made by the mind, and the part is posterior to the whole. In real things, on the contrary, units are prior to the multitude, and multitudes only exist through units. (The same holds of changes, which are not really continuous.)[17]

[17] Quoted in Russell, *A Critical Exposition of the Philosophy of Leibniz*, 245.

Leibniz thus held that "within the ideal or continuum, the whole precedes the parts" and the parts are potential; on the other hand, regarding "real things," the parts are always given actually and prior to the whole. Leibniz appears to have thought, at one point, that such considerations sufficed "to dispel the difficulties regarding the continuum—difficulties which arise only when the continuum is looked upon as something real, which possesses real parts before any division as we may devise, and when matter is regarded as a substance."[18]

Leibniz's "law of continuity," that nature does not make leaps, is applied above all in physics, where he claims it has "a great use" in "destroying atoms, small lapses of motion, globules of the second element, and other similar chimeras."[19] Moreover, on this account,

> matter would be divisible everywhere and more or less easily with a variation which would be insensible in passing from one place to another neighbouring place; whereas, according to the atoms, we make a leap from one extreme to another, and from a perfect incohesion [...]. And these leaps are without example in nature.[20]

Leibniz further elaborates on his principle or law of continuity, showing that it has implications not just on the whole of physics but even on a proper theory of perception and mind

[18]Quoted in Russell, *A Critical Exposition of the Philosophy of Leibniz*, 245. But Leibniz did also endorse an actual infinity:

> I am so much for the actual infinite that instead of admitting that nature abhors it, as is commonly said, I hold that it affects nature everywhere in order to indicate the perfections of its Author. So I believe that every part of matter is, I do not say divisible, but actually divided, and consequently the smallest particle should be considered as a world full of an infinity of creatures [...]. (Leibniz and Wiener, *Leibniz. Selections. Edited by Philip P. Wiener*, 99)

[19]Ibid., 71.

[20]Quoted in Russell, *A Critical Exposition of the Philosophy of Leibniz*, 235.

(*pneumatique*), a connection that helps better explain his particular treatment of continuity as "ideal." This law

> entails that one always passes from the small to the large and back again through what lies between, both in degrees and in parts, and that a motion never arises immediately from rest nor is it reduced to rest except through a lesser motion, just as we never manage to pass through any line or length before having passed through a shorter one. [. . .] All this can allow us to judge that noticeable perceptions arise by degrees from ones too small to be noticed. To judge otherwise is to know little of the immense subtlety of things, which always and everywhere involves an actual infinity.
>
> I have also noticed that because of insensible variations, two individual things cannot be perfectly alike but must always differ in something over and above number.[21]

These virtual (*virtuel*) *mikron*, or imperceptible changes, are one of the ways Leibniz continues to bring to bear arguments against the existence of atoms:

> Moreover, there are a thousand indications that allow us to judge that at every moment there is an infinity of perceptions in us, but without apperception and without reflection—that is, changes in the soul itself, which we do not consciously perceive [*appercevons*], because these impressions are either too small or too numerous, or too homogeneous, in the sense that they have nothing sufficiently distinct in themselves; but combined with others, they do have their effect and make themselves felt in the assemblage, at least confusedly. It is in this way that custom makes us ignore the motion of a mill or of a waterfall, after we have lived nearby for some time. It is not that this motion ceases to strike our organs and that there is nothing corresponding to it in the soul, on account of the harmony of the soul and the body, but that the impressions in the soul and in the body, lacking the appeal of novelty, are not sufficiently strong to attract our attention and memory, which are applied only to more demanding objects. [. . .] In order better to recognize [*juger*] these tiny perceptions [*petites perceptions*] that cannot be distinguished in a crowd, I usually make use of the example of the roar or noise of the sea that strikes us when we are at the shore. In order to hear this noise as we do, we must hear the parts that make up

[21] "Preface to the New Essays on the Understanding," in Leibniz, *Philosophical Essays*, 297.

> this whole, that is, we must hear the noise of each wave, even though each of these small noises is known only in the confused assemblage of all the others, and would not be noticed if the wave making it were the only one. For we must be slightly affected by the motion of this wave, and we must have some perception of each of these noises, however small they may be, otherwise we would not have the noise of a hundred thousand waves, since a hundred thousand nothings cannot make something. Moreover, we never sleep so soundly that we do not have some weak and confused sensation, and we would never be awakened by the greatest noise in the world if we did not have some perception of its beginning, small as it might be, just as we could never break a rope by the greatest effort in the world, unless it were stretched and strained slightly by the least efforts, even though the slight extension they produce is not apparent.[22]

It follows that these "petite perceptions" are "more effectual than one thinks":

> They make up this I-know-not-what, those flavors, those images of the sensory qualities, clear in the aggregate but confused in their parts; they make up those impressions the surrounding bodies make on us, which involve the infinite, and this connection that each being has with the rest of the universe. It can even be said that as a result of these tiny perceptions, the present is filled with the future and laden with the past, that everything conspires together (*sympnoia panta*, as Hippocrates said), and that eyes as piercing as those of God could read the whole sequence of the universe in the smallest of substances.[23]

The overall idea, in short, is that continuity is not just an idealization reflecting the nature of our perceptions as "clear in the aggregate but confused in their parts," where we must pass over the many distinct "micro-perceptions" that are "too small to be noticed," but is a useful hypothesis in that it guarantees that, even if we cannot discriminate all the intermediate degrees in a change, in truth we "always pass from the small to the large and back again through what lies between." This is a way of further guaranteeing that each being

[22]Leibniz, *Philosophical Essays*, 295-96.

[23]Ibid., 296.

has a “connection” (or “conspires together”) with the rest of the universe. In other words, imagining nature is continuous, even though no single perception allows us to draw such a conclusion, allows us to ensure the *causal connectivity* of nature and its changes, i.e., that “the future is laden with the past” and “everything conspires together.”

A potential problem with this general approach—that holds that the resolution limit of our perceptive capacities does not reveal a fundamental discreteness, but that we should *assume* continuity as an “ideal” limit of resolution or as a simplification arising as some sort of “compression” of the massive (even infinite) number of distinguishable “micro-perceptions” contained in any one perception—is that there are others who use this very same sort of reasoning, but different guiding examples, to argue to the exact opposite conclusion (that nature is “really” continuous, and our discrete appropriations of it are simplifications and idealizations). For instance, René Thom says:

> The discrete character of a transformation is a simplification created by our organs of perception. We are essentially designed to see discontinuities. They alone have meaning. It is essential to an animal that it recognize its prey: It must be both recognized and localized. Therefore, there have to exist mechanisms in the nervous system that make possible to instantaneously discriminate between what’s living and what isn’t. Among the criteria required for this discrimination must figure the identification of the discontinuities and general contours of the object. Then there are activities, such as human language, which assume this discretization: Spoken language is formed from discrete phonemes. Yet underneath these phenomena, at their foundations, lies something continuous. Although a Fourier spectrum can be very complicated, it can be continuously modified with the help of a sound synthesizer. The sound of “B” can be continuously changed into the sound of “P” by a simple transformation. But when someone listens to them he knows right away “This is a ‘B’, that is a ‘P’”. He will perceive a complete

> discontinuity between the two sounds; he will not be able to perceive the continuous transformation.[24]

Frequently, the dispute between these two positions—that continuity is a "simplification" or "ideal limit" of a massive number of really discretized items and that discreteness is an "ideal" or "simplification" of what is really continuous—will settle on a "compromise" position, reverting to some form of endorsement of the situation-dependence or scale-dependence of the notions of continuity and discreteness.

Stepping back, then, Randomness and Idealization, in admittedly different ways, both appear to believe that there is an alliance between continuity and "larger" scales (involving an aggregation of a large number of components). In regarding a large number of the components of a system and locating continuity in either the "average behavior" of such large numbers or in "simplifying" or "clarifying in the aggregate" the underlying complex interactions between the many discrete components "confused" in their parts, an alliance is forged between continuity and a large number of components (and the sorts of "average behaviors" characteristic of such scales). In terms of generality, then, one could argue that this approach—in characterizing continuity-through-large-numbers (or "large scales")—holds that the "larger" the level on the scale, and the more components involved in the system, the more general the emerging features or behavior may be. In short: when continuity is held to be a matter of what happens over larger scales (in relation to what happens at smaller or more local scales), the measure of greater generality is also held to be a matter of behavior or features that emerge over (or obtain for) larger and larger scales.

[24]Thom and Noël, *To Predict is Not to Explain*, 79-80.

Relation of Parts

Density

Here a continuum is taken to be an aggregate of simples (instants or points or some other such "point-like" objects) so arranged in relation to one another that they have the property that between any two (simples, parts or points) another can be found, i.e., "each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbor."[25] In other words, a continuum is an infinite aggregate of simples (points) satisfying the property of *density*.

A major problem with this account is that density is in fact insufficient to guarantee continuity, as is easily demonstrated by the fact that the ordered set of rational numbers are dense but not continuous (in really any useful sense, but also in an important and precise sense explored in the section on Arithmetization).

Compositional

The next three sub-characterizations under the heading of *composition* share the assumption that continuity-discreteness is about particular relationships between parts and their wholes.

Self-Similarity

In some sense, this is a way of using the concepts found in the "scale" approach to develop a different characterization of continuity as a particular type of *invariance*, namely invariance against certain changes in scale or size. One particularly powerful (but less restrictive) way

[25]Bolzano, *Paradoxes of the Infinite*, 129, cited in Bell, *The Continuous and the Infinitesimal in Mathematics and Philosophy*, 134.

this gets developed, and also connects with a distinct conception of generality, is in terms of Spinoza's ideas of common notions as capturing, at the level of ideas, that which, at the level of bodies, amounts to an invariance in the patterns of communicating motions (in *one* body and another *one* body) that is also *equally in the part and in the whole* (formed by the two bodies now composing to form another *one*).[26]

Arithmetized: Weierstrass-Dedekind-Cantor

The main idea here is to *reduce* the continuous to the discrete, or perhaps to "imitate" the continuous within fundamentally discrete structures. Here, continuity is held to be embodied by "the continuum" and this is constructed by gradually building upon finite, discrete entities (like the integers). This construction usually proceeds by means of various "succession" functions—ways of moving from existing elements to new elements, step by step, generating an entire class of objects from a simple base—up to the point that *nothing more can be added*. In other words, continua are assembled by simple iterative procedures from isolated points or basic units that are taken to be primary or already given. Having built up the continuum, cutting the resulting continuum anywhere you like will necessarily put you at a *definite element* of the completed continuum, not *indefinitely between* entities—and it is this fact that provides one way of characterizing continuity.[27] Building on Weierstrass's initial rigorous arithmetical definition of real numbers and his more general approach to eliminate any reliance on the intuitions of continuous motions from analysis, Dedekind constructs

[26]See Chapters 4-5, especially Chapter 5, for more details.

[27]This general approach often makes its non-intuitive, non-geometrical, flavor a point of pride. 19th century mathematicians like Weierstrass (with their efforts to construct the real numbers from the integers and rationals) fall into this category.

the real numbers through Dedekind cuts. Cantor takes this still further, considering the continuum in terms of continuous *sets* with the two properties of being both connected and perfect. A set S is perfect if it is equal to the set of all its limit points (its "derived set"), i.e., if $S = S'$, where S' is the set of all limit points of S. A connected set is one that cannot be partitioned into two non-empty subsets such that each subset has no points in common with the set closure of the other (where the set closure of a set is the smallest closed set containing that set, i.e., the set plus its limit points).[28]

Note that this definition first specifies certain basic elements (points), and then requires that, in *composing* the continuum out of these basic elements, the connections between the points of the set be of a certain sort. With this approach, continuity is not really a property of the real numbers themselves, so much as the result of aspects of the structures that emerge by requiring the points to relate in a certain way.[29]

One objection to these sorts of "point continuum" approaches to continuity could be that, in making discrete elements (namely points) logically prior to the continuum itself as well as forming parts of it, a feature that one might arguably like to attribute to the continuum, namely that the parts of a continuum are themselves continuous, must be abandoned. Other objections to the general idea of this approach seem to share a similar form, which looks something like this: while Greek mathematics would create a fateful division between the arithmetic and the geometrical, in the modern era (since at least the 17th century),

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[28]In these terms, Dedekind's set of real numbers has the property of perfection (but not connectedness). This notion of perfect sets is similar to one of Peirce's earlier descriptions of the characteristics of a continuous set as containing all of its limit points (in addition to being, on his account, non-denumerably infinite and dense).

[29]Note also that the set-theoretical identification of the continuum with the Cantorian real number line, constructed out of "points," forges a strong connection between the uncountably infinite and continuity.

nearly every sufficiently powerful concept in math has straddled the divide between the two. Because of the vital role of the continuum in so much of math, then, an adequate conception of the continuum, as an extremely powerful schema serving as the site for a number of concepts and techniques, should accordingly not be reduced to either side of the polarity, but should embrace both the arithmetic and the geometric. Of course, Cantor's approach can be seen as showing that the geometric linear continuum is isomorphic with the arithmetic continuum (with the consequence that, due to the Archimedean nature of the real number system, infinitesimals are superfluous). But the objection is that while the Cantor-Dedekind approach might have succeeded in bridging the gap between arithmetic and standard Euclidean geometry, it fails to provide an appropriately richer theory of continua that embraces infinitesimals, allows for generalizations of Cantor's approach to the infinite, and establishes a setting for other non-Archimedean ordered algebraic and geometric systems.[30]

Connectedness

Above, I mentioned the well-known result from analysis that the continuum (and its closed intervals) can be shown to be characterized as *connected* in the sense that it cannot be split into two non-empty subsets neither of which includes a limit point of the other. A stronger version of connectedness is given by *indecomposability* and, in certain settings, the continuum can be shown to be indecomposable, where this means continua cannot be split *in any way*

[30] See, for instance, Ehrlich, "The Absolute Arithmetic Continuum and the Unification Of all Numbers Great and Small," for a use of *surreal numbers* to repair some of these deficiencies and for an argument that "whereas the real number system should be regarded as constituting an arithmetic continuum modulo the Archimedean axiom, the system of surreal numbers (henceforth, **No**) may be regarded as a sort of absolute arithmetic continuum modulo NBG (von Neumann-Bernays-Gödel set theory with global choice)" (Ehrlich, 3). In the Size section below, I discuss surreal numbers in a little more detail.

whatsoever into two or more parts or sections having nothing in common. It of course follows from this approach that *continua cannot be composed of their parts* (so this belongs in the general "compositional" framework only in the sense that it is characterized as a negation of compositionality). This approach can be defended within the context of Smooth Infinitesimal Analysis (SIA), in which context all functions between the objects of the relevant category are smooth (differentiable arbitrarily many times) and thus continuous, and so the concept of *continuity* emerges as primary, no longer derived from or explained in terms of the discrete. The basic object (space) of any smooth world $\mathbb{S}$ is an indefinitely extensible homogeneous straight line R, called the smooth (or real) line. The basic axioms describing $\mathbb{S}$ do not, importantly, exclude the possibility that in R we may have $x^2 = 0$ without being able to affirm that $x = 0$. Thus, if we then define the part $\Delta \subset R$ as consisting of those points x for which $x^2 = 0$, i.e.,

$$\Delta = \{x \mid x^2 = 0\}, \tag{7.1}$$

then we can assert the possibility that Δ does *not* reduce to $\{0\}$, which implies that R cannot be considered equal to the usual set-theoretic field $\mathbb{R}$.[31] In this context of smooth worlds, it is easy to demonstrate the main result concerning how a connected continuum-object R is continuous in the very strong sense of being indecomposable or non-detachable.

Definition 7.0.1. We call a part U of R *detachable* if, for any x in R, it is the case that either x is in U or x is not in U. Put otherwise: a part U is detachable if there exists a complementary part V of R such that U and V are disjoint but together cover R.

[31] Δ is also sometimes called the (basic) *microneighborhood* (of 0), or the object of *nilpotent infinitesimals* (of square 0).

Then it can be proven that

Theorem 7.0.1. The only detachable parts of R are R itself and its empty part.[32]

In $\mathbb{S}$, the smooth line R can be seen to be *non-detachable* (or *indecomposable*) in the sense that it cannot be split *in any way* into two disjoint non-empty parts, a result we can extend to any closed interval of R (and, in fact, thereby, to any interval in R).[33] This result might also be thought of as revealing that in smooth worlds in general, truth values cannot be merely two-valued. In fact, in this context the law of excluded middle *must* fail in general; and in a sense it is the underlying intuitionistic logic that enables the possibility of the non-degeneracy of the infinitesimal microneighborhoods.[34] Physically interpreted, this non-degeneracy can be regarded as stipulating that *the instant (in a movement) cannot be reduced to a point.* Given the fact that the elements of R satisfy the law of excluded middle for closed sentences but that in general the law of the excluded middle is refutable in SIA,

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[32]See Bell, *A Primer of Infinitesimal Analysis*, 29-30, for a proof of this theorem.

[33]See ibid., for a brief and very accessible introduction to SIA, and for a demonstration of the parenthetical remark above. The interested reader can also see Moerdijk and Reyes, *Models for Smooth Infinitesimal Analysis* and Kock, *Synthetic Differential Geometry*, the standard references in this field. As for the definition of indecomposable, the purpose of saying "in any way" is to highlight that this is a much stronger result than the result in analysis that claims that the continuum and its closed intervals are *connected* in the sense that they cannot be split into two non-empty subsets neither of which includes a limit point of the other.

[34]However, in most models of SIA, the law of excluded middle will be true in the restricted sense, namely whenever α is a *closed* sentence (having no free variables), then indeed $\alpha \vee \neg\alpha$ will hold. See McLarty, *Elementary Categories, Elementary Toposes* for details. As Bell notes: "Thus, in smooth infinitesimal analysis, the law of excluded middle fails 'just enough' for variables so as to ensure that all maps on R are continuous, but not so much as to affect the propositional logic of closed sentences" (Bell, *A Primer of Infinitesimal Analysis*, 106). This is especially curious because the well-adapted models of SIA still do not allow us to go on to infer from the fact that the law of excluded middle applies to arbitrary individuals/points that the corresponding universal generalization, i.e., *for all* x and *for all* y in R either $x = y$ or $x \neq y$, holds. In fact, we know that the latter statement is positively refutable in SIA (for the identity relation on R is not decidable). Certain elements of R simply cannot always be distinguished, though it does also contain points that can be distinguished; however, on account of the existence of the former, as Bell notes, R *cannot* be thought of as the sum total of its elements.

one might also think of such elements as "potential" (see the section below on Possibility). In this context, potentiality could be further thought of as *enriching* the continuum.

Ignoring some of these interesting connections (with potentiality and the failure of the law of the excluded middle), the main take-away from this characterization should be that continua (embodied in the smooth line) cannot be composed of their parts, or in any way split or detached into two disjoint non-empty parts.

Reflexivity

This is a property held to characterize a continuum by many diverse thinkers, from Aristotle, to Kant, to Charles Peirce. Peirce articulated this idea when he said that

> a continuum is that of which every part has itself parts of the same kind.[35]

Zalamea names this property *reflexivity*.[36] As Peirce would come to realize, reflexivity is not the same thing as infinite divisibility, but rather implies that a continuum cannot be composed of points (what Zalamea calls its inextensibility) and, strictly speaking, is not even *divisible*, but only contains points when "the continuity is broken by marking the points."[37] Thus, on this account, a continuum, precisely *as* continuous, "contains no definite parts; its parts are created in the act of defining them and the precise definition of them

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[35] "The *Century Dictionary* Definition of 'Continuity' (Note 5), in Peirce, *Philosophy of Mathematics*, 138.

[36] See Zalamea, *Peirce's Logic of Continuity*, 16-18.

[37] Peirce, *Philosophy of Mathematics*, 138. That reflexivity is not the same thing as infinite divisibility is as it should be, for we know, as Peirce himself notes (at CP 6.168), that the series of rational fractional values is infinitely divisible but not regarded as continuous by anyone.

breaks the continuity."[38] Contraposing the statement that reflexivity implies inextensibility, we could equivalently say that the extensibility of a continuum implies its irreflexivity, which has implications for the arithmetized conception of the continuum as an extensible sum of points. In short, on this account, saying that every part has parts of its own "of the same kind" ultimately entails that the continuum is not "really" composed of (non-continua-like) points or divided into simple parts at all.

Regularity

This is a very general characterization, imposing minimal requirements on the form of a continuum. It simply stipulates that continuity is "a certain kind of relationship of each part to all the coordinate parts."[39] In principle, then, without further specification of the nature of this relationship, we only require that there exist a certain *regular* relationship between each part and all the other parts.

Homogeneity

This is, of all the models falling under the heading of Relationships of Parts, perhaps the "weakest" requirement (in the sense that it puts the fewest constraints on the form taken by the continuous object). Thom articulates this when he speaks of the "archetypal continuum" as a "space which possesses a perfect qualitative homogeneity," possessing "no structure by itself (whether metrical or simply differential): the only demanded property is its qualitative

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[38] Peirce, *Philosophy of Mathematics*, 138-139.

[39] "Some Amazing Mazes—Supplement," Peirce, *Collected Papers of Charles Sanders Peirce*, 7.535 note 6.

homogeneity."[40] This characterization is arguably ultimately identical to the reflexivity of the continuum, so it is no accident that, like Peirce, Thom argues that the discrete and anything point-like represents only an "intrusion" by means of a "cut" in the continuum.[41] More generally, this "qualitative homogeneity" can also be compared to Aristotle's alignment of continuity with *suggenicity*.[42]

Contiguity/Positional

Beginning with Aristotle, continuity is "something of the contiguous," i.e., is a subclass of contiguous things (which is itself a subclass of successive things). Continuity is found when two contiguous elements have "become one," glued together as a unity due to the fact that the limit of each, along which they touch, has become one and the same limit.[43] This is grouped under Relationships of Parts, for Aristotle insists that the manner in which something becomes a unity as continuous (articulated in terms of the behavior of "touching" parts) determines the manner in which it becomes a *whole*.

Issue of Distinction

Indistinctness

Here we have a kind of merging of one part into another, enabling passages from one realization or expression of the properties of a continuum to another by insensible degrees. An

[40] Thom, "L'Anteriorité Ontologique du Continu sur le Discret", in Salanskis and Sinaceur, *Le Labyrinthe du continu*, 141.

[41] See ibid., 142.

[42] See Chapters 1 and 2.

[43] There are other aspects to Aristotle's complete definition of continuity, such as suggenicity (and the related fact that the "fused" parts become "impassive" to one another); see Chapter 2 for details.

example is provided by the color wheel. The constituent individuals or parts cease to retain their distinct and independent existence, but rather only have existence in their relations to one another. All individual parts are "fused" into one another, but here the emphasis is on the *lack of distinction* between the individual components, *as individuals*, of the continuum.

Fuzziness

Continuity describes objects (or classes, qualities, quantities, intervals, even truth values) the parts of which belong or inhere with variable degrees. The boundaries of a given whole are accordingly not exactly defined; there are no discrete points or edges to demarcate *crisp* boundaries. Interestingly, admitting this initial imprecision allows for far finer "internal" resolution and more powerful and flexible representations of complex, dynamic systems more generally. The underlying logic must be non-classical, and the size of any given collection (viewed "internally") will be infinite. If we restrict to intensive quantities, in general such a quantity cannot be determined by the total of its values at *points*.[44]

Normally, in classical mathematics, the transition for an element or part between membership and nonmembership in a given quality (or set) is abrupt and crisp. By contrast, without explicitly questioning the usual excluded middle axioms, both Scotus and Oresme clearly treated qualitative alteration as involving intermediary states and continuous changes in degree. It is thus entirely natural to speak of Scotus and Oresme as thinking in terms of qualities or multitudes as being characterized, "intrinsically," by a continuum of grades of membership. In accord with this aim, we can produce measurements of the membership

[44]Some of the Stoics appear to have entertained aspects of this model; it achieves perhaps its most elaborate articulation in fuzzy logic.

of a part or element in a set (or in some form or quality) by functions that capture both gradual and uneven continuous alterations, and we could thus informally define a *Scoresme set* (for Scotus and Oresme, obviously)—or, in another context, a fuzzy set—as a multitude containing parts or elements with varying degrees of membership. Following the lead of fuzzy logic, parts of a Scoresme set will be mapped to membership values using functions. We can think of the functions as mapping parts of a Scoresme set to a real number on the interval $[0, 1]$. Full membership of any element x is 1; no membership is 0.[45] Everything in between is available to us.

Consider the following example sets (technically discrete and finite for simplicity, but still perfectly capable of illustrating the idea of partial, continuously variable, membership):

- $I = \{I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII\}$: Intensity of earthquakes, on the universe of earthquakes, given by the Mercalli scale
- $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$: Hardness of mineral, on the universe of minerals, given perhaps by the Mohs scale
- $F = \{1500, 2175, 7000, 12750, 16500, 20000\}$: Fracture strengths, on the universe of clay bricks, given in units of pounds per square inch
- $C = \{\text{red, green, blue}\}$: Basic color composition, on the universe of colors
- $R = \{\text{unripe, maturing, ripe}\}$: "Ripeness," on the universe of fruits
- $D = \{ppp, pp, p, mp, mf, f, ff, fff\}$: Dynamics or "loudness," on the universe of musical objects
- $A = \{\text{newborn, young, middle-aged, old}\}$: Age, on the universe of human beings

[45] Restricting ourselves to these two extremes would put us back in the classical realm.

Then, incorporating membership values for some of these, we might get:[46]

- $I = \left\{\frac{0.1}{V} + \frac{0.6}{VI} + \frac{1.0}{VII} + \frac{0.8}{VIII} + \frac{0.2}{IX}\right\}$: indicating an earthquake of intensity "around 7," based on averages of readings from various locations
- $C = \left\{\frac{0.6}{\text{red}} + \frac{0.75}{\text{green}} + \frac{0.1}{\text{blue}}\right\}$: A particular shade of yellow
- $R = \left\{\frac{0.9}{\text{unripe}} + \frac{0.4}{\text{maturing}} + \frac{0.05}{\text{ripe}}\right\}$: A barely-edible avocado
- $D = \left\{\frac{0.2}{p} + \frac{0.56}{mp} + \frac{0.49}{mf}\right\}$: A not-too-soft 'piano'
- $A = \left\{\frac{0.66}{young} + \frac{0.34}{middle-aged}\right\}$: A 25-year old.

In fuzzy set theory, membership is determined by functions. For a simple, intuitive example, consider the following membership function for "young" with respect to the set A from above:

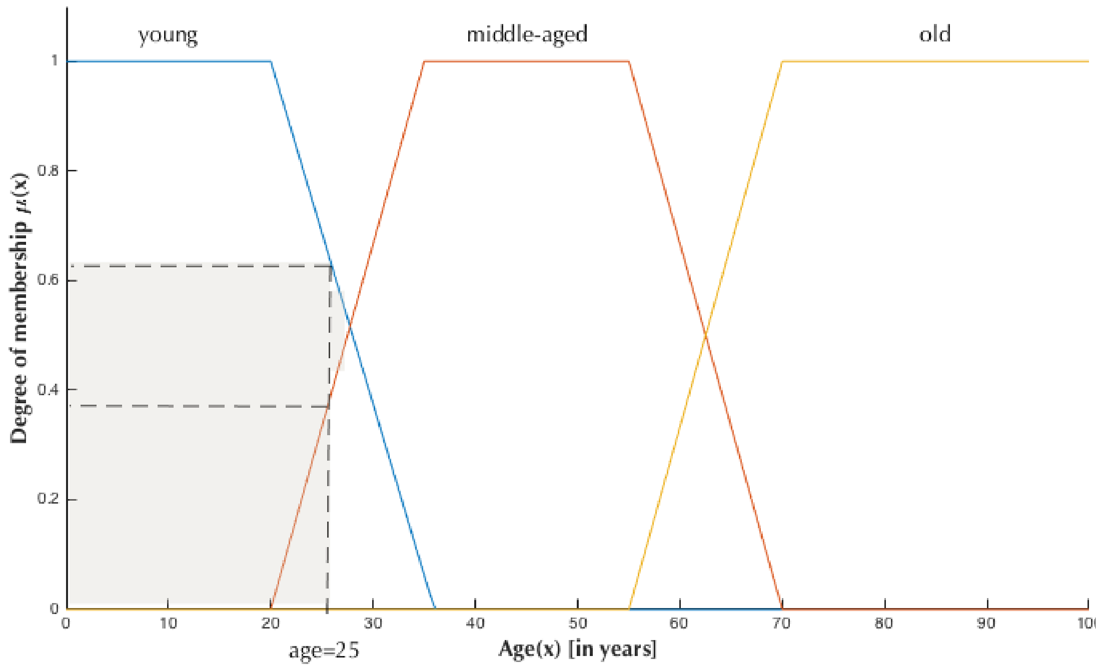
$$\mu_{young}(x) = \begin{cases} 1 & age(x) \leq 20 \\ 1 - \frac{age(x)-20}{15} & 20 < age(x) \leq 35 \\ 0 & age(x) > 35 \end{cases} \tag{7.3}$$

Constructing similar functions for the other terms of the age set, we might represent a 25-year old ($\left\{\frac{0.66}{young} + \frac{0.34}{middle-aged}\right\}$) in terms of figures, following Oresme's lead and also the standard development of fuzzy set theory, as follows:

[46] In terms of symbolic notation, when the universe of discourse, X, is discrete and finite, a Scoresme set S might be denoted, following Zadeh, *Fuzzy sets, fuzzy logic, and fuzzy systems*,

$$S = \left\{\frac{\mu_S(x_1)}{x_1} + \frac{\mu_S(x_2)}{x_2} + \cdots\right\}. \tag{7.2}$$

The idea with this notation is that the bar is merely a delimiter or separator and does *not* represent a quotient. Similarly, the numerator is simply the value associated with the membership of some part in the set S, while the denominator is just meant to index that membership value to the appropriate element. Also, the summation symbol does not represent algebraic summation, but acts instead as an aggregation operator.



In this context, one can then further develop *relations* between fuzzy sets, and measure the "strength" of relations between ordered pairs of the two universes with a membership function expressing degrees of strength of the relation. All the standard operations on classical sets—like Set Difference and De Morgans's laws—hold for such sets, with one exception, namely the excluded middle axioms.

In short, continuity is regarded as a matter of "intensive" variability within a given range, where the form or quality or set subject to such intensive variability is regarded as admitting of intermediate degrees of realization or gradual changes in partial membership, as opposed to the classical abrupt or crisp transitions between one state and its opposite or the strict dichotomy between membership and non-membership. In this way, a more nuanced account can be given of all kinds of gradual and progressive changes, and a vision emerges of various objects and phenomena as simultaneously inhabiting a number of overlapping, partially compatible states. It is also a more natural way of dealing with the fact that with certain changes, such as a person growing older, it is not that we do not yet know the precise cut-off between young and not young (or old), but that the sorts of changes involved in one instant or minute or hour in the passage of a person's life do not seem to be the sort of thing that could provide a foundation for a difference between being a child or young person and being an adult or no longer young. On the other hand, there are trade-offs, and even some of the features of fuzzy logic remain problematic in relation to certain intuitions: for instance, by fuzzifying the truth-values and making truth a matter of degrees, in a sorites-type transition there must still be some point where the truth value changes from completely true to *less than completely true* (though it is not clear whether the existence of such a point is any more intuitive than non-intuitive crisp changes).[47]

Possibility

The idea here goes back to Aristotle, and involves an alignment between continuity and possibility or potentiality, on the one hand, and discreteness (or cuts in the continuum) and actuality on the other. There are two main ways this is developed, again going back to Aristotle. The first refers to Aristotle's discussion of the continuum in terms of the indefinite potentiality for division. The second way of developing this refers to Aristotle's various discussions, themselves less appreciated, to the effect that there are definite "potencies" that characterize a continuous entity:

[47]See Priest, *An Introduction to Non-Classical Logic, Second Edition*, Chapter 11, for a discussion of some of these issues.

> all states [ἕξεις] in virtue of which things are altogether impassive [ἀπαθῆ ὅλως] to change or unchanging, or are not easily changed for the worse, are called potencies [δυνάμεις]. For things are broken and crushed and bent and in general destroyed, not because they have a potency, but because they do not have one and are deficient in some way. And things are impassive [ἀπαθῆ] to such processes when they are hardly or slightly affected [πάσχει] by them because they have the potency [δύναμιν] and the ability [δύνασθαι] to be in some definite state [τῷ ἔχειν πώς]..[48]

In this context we see that such "impassiveness" to certain changes, or *invariance* (either completely or "not easily for the worse")—something specifically attributed to what is continuous—is precisely aligned with the having of definite potencies rendering one "immune" to certain changes and destructions. Whitehead's notion of the "extensive continuum," in which context he draws the distinction between 'general potentiality' and 'real potentiality', belongs here as well, where the latter is "conditioned by the data provided by the actual world...and is relative to some actual entity," while the former is "absolute" and "expresses the solidarity of all possible standpoints throughout the whole process of the world."[49] For Whitehead, the reality of the future (and its connectedness with the past) is bound up with these real potencies, and these must be considered in their "character of a real component of what is actual."[50] On the other hand, in the more general sense, the continuum is just the "potentiality for division," while "an actual entity effects this division."[51] This particular development of the notion of continuity via potentiality could be thought of as attempting to unify both of the Aristotelian approaches to potentiality, or both the standard

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[48]Aristotle, *Metaphysics*, 1019a27-30.

[49]Whitehead, *Process and Reality*, 65-66.

[50]Ibid., 66.

[51]Ibid., 67.

Aristotelian notion of an "abstract" *potentiality for division* as well as the more dynamic and non-epistemic sense in which "definite atomic actualities determine one coherent system of real divisions" which contribute to the formation of "real potentialities whose solidarity the continuum expresses." Adopting his technical vocabulary, the extensive continuum (as potentiality) is the site or locus sponsoring the *concresence* and *objectifications* of actual or individualized entities:

> This extensive continuum is one relational complex in which all potential objectifications find their niche.[. . .] It is not a fact prior to the world; it is the first determination of order—that is, of real potentiality—arising out of the general character of the world. In its full generality beyond the present epoch, it does not involve shapes, dimensions, or measurability; these are additional determinations of real potentiality arising from our cosmic epoch.
>
> [. . .] In the mere continuum there are contrary potentialities; in the actual world there are definite atomic actualities determining one coherent system of real divisions throughout the region of actuality. Every actual entity in its relationship to other actual entities is in this sense somewhere in the continuum, and arises out of the data provided by this standpoint. But in another sense it is everywhere throughout the continuum; for its constitution includes the objectifications of the actual world and thereby includes the continuum; also the potential objectifications of itself contribute to the real potentialities whose solidarity the continuum expresses. Thus the continuum is present in each actual entity, and each actual entity pervades the continuum.[52]

An actual entity's "becoming" depends upon its "realization of a proper region" within a continuum, which in turn depends upon determining its boundary within this relational complex; but the extensive continuum "in itself" remains without boundary, while not for all that being undifferentiated.

[52]Whitehead, *Process and Reality*, 66-67.

In this context, one could also mention both Peirce's notion that "The continuum is concrete, developed possibility,"[53] and how, in the context of fuzzy set theory, fuzzy sets can be developed in terms of "possibility measures" and "possibility distributions" for the quantity varying with respect to membership in some attribute.[54]

Structural

The idea here is that continuity is not a matter of points and their relations of, e.g., distance, or some "regularity" in their relations, but rather a matter of certain structural characteristics that remain invariant or are preserved through change or in passing from one system to another.

Functorial (Structure-Preserving)

In the context of category theory, as discussed in Chapter 6, a *(co)continuous functor* can be defined as a special sort of functor (structure-preserving action) that takes universal objects in the source category to universal objects in the target category. As such, it can be thought of as follows: whatever else the functor does to objects as it takes objects from one category to another, a (co)continuous functor will send the object that acted as a privileged gateway or intermediary ((co)limit) in relation to the rest of the objects in the source category to an object that similarly plays the role of privileged intermediary for its fellows in the target category. In other words, continuity is a special kind of passage or translation from one "world" of objects to another "world" of objects—*special* in that, in passing from one

[53] Peirce, *Philosophy of Mathematics*, 176.

[54] See Ross, *Fuzzy Logic with Engineering Applications*, Chapter 15, for more details; and Zadeh, "Fuzzy sets as a basis for a theory of possibility."

"world" to the other, it takes care to preserve the special role of those unique objects that act as privileged intermediaries for the rest of the objects of their world, forging a direct line of communication between those special objects.

Agreement/Concord

On the continuum of sound, the closest notes—beyond the "just noticeable difference," where the two are first detectable as non-identical—will usually be *the most dissonant* or the "furthest" from one another in terms of any structural conformity between the ratios of intensities. On the other hand, the mutual agreements and more composite unities produced by certain combinations of notes, are forged along the lines of some structural conformity between the ratios of intensities characteristic of each separate sound—something that has nothing to do with closeness in the sense of proximity. Those sounds that combine well with other sounds do so on account of the greater conformity between their ratios of intensity of that quality, and *not* on account of any uniform relationship of *closeness* imposed on the continuum of the quality (sound) from without. In speaking of intensities figured with continuous geometrical figures, and in developing the relations between these in terms of ratios of such intensive magnitudes, Oresme extended these features of the musical continuum to continua more generally, allowing him to advance a concept of continuity freed from closeness and uniform distance metrics. Continuity is thus to be found in morphological agreements or structural conformities that obtain between distinct intensities and their distinct manners of variation. The power of this idea is due, in large part, to Oresme's bold attempt to "figure" all kinds of intensities and qualities, so that the "accords" between magnitudes can be extended to an analysis of nature more broadly—ultimately leading to a theory of nature as being differentiated along lines of "natural friendship" and "natural hostility," itself unpacked in terms of agreements and disagreements between the ways the forms and qualities that different beings support change in intensity.

Cohesion

We typically think of "cohesion" in terms of topology, where cohesion is something like the specification of how points or objects in a space "hang together." But many different contexts (categories) present us with differing modes of cohesion and variation. Since in the context of category theory, functors sometimes allow us to relate and compare categories, the question of which settings for modeling objects are more cohesive/variable and which are more discrete/constant should involve functorial comparisons. One would like to more precisely study and control the contrast between the degree and type of continuity (*qua* cohesiveness/variability) or discreteness (*qua* non-cohesion/constancy) of these related but distinct settings. Geometric morphisms between toposes enable us to begin to compare the manner in which objects living in different environments (categories, toposes) hang together or cohere—giving rise to a "science of cohesive toposes."[55] In short, then, continuity is developed in terms of the particular way objects of a certain sort "hang together" as compared to the manner by which objects of other sorts "hang together."

In a very different context, Aristotle's positive formulation of continuity, in which what holds the parts of a continuous thing together and "fuses" them takes precedence over

[55]For more on this, see Chapter 6.

its (accidental) capacity to be divided and have its parts "torn apart," could fall under this heading.

In terms of generality, while each of the above sub-characterizations of Relation of Parts develops somewhat distinct connections to generality, all of them are joined together by the fact that they seem to hold generality to be a feature that belongs above all to the *relations* between parts of a whole, i.e., to be a matter of relationality.

Closeness

This general approach embraces both Aristotle's early alignment of continuity and closeness and the standard ϵ-δ definition of continuity (from calculus) in terms of "degrees of closeness." If you imagine a function as a measurement of some sort taken at points (of some variable, such as time, or in space), the basic question here is "do samples of points that are 'close' have corresponding 'similar' or 'close' measured function values?" In other words: "does it ever happen that over very small distances, the values assigned to the respective points are *not* very close or similar to one another?" A little more formally: if you have a function f on the real numbers, the idea is to decide which conditions the function must satisfy in order for us to be able to say "the function f is continuous at a point $a \in \mathbb{R}$," and to develop such conditions in the form of a precise formulation of the statement "a value $f(x)$ will be close to the value $f(a)$ whenever the point x is close to the point a." Of course, there is a standard distance function for the real numbers, so we already have a measure of the degree of closeness of two numbers. However, the question arises: just *how close* must the function evaluated at x be to the function evaluated at a? The fundamental idea then comes by realizing that instead of specifying a particular degree of closeness (of $f(x)$ to $f(a)$), we can require that *no matter what* choice we make for the degree of closeness (no matter how small), it can be arranged so that $f(x)$ is within this prescribed degree of closeness to $f(a)$, whenever x is within some corresponding degree of closeness to a. This can further be described in terms of *limits*, where this is the value a function or sequence "approaches" as the input "approaches" some value.

This well-established approach to continuity in terms of closeness is arguably realized in the most powerful way with the full use of infinitesimals and the corresponding notion of "micro-continuity." The full use of infinitesimals allows for a non-punctual conception of the continuum, building on the intuition that a continuum does not contain points but can have the infinitesimals as (non-point-like) parts of the continuum.

> Traditionally, an infinitesimal quantity is one which, while not necessarily coinciding with zero, is in some sense smaller than any finite quantity. In 'practical' approaches to the differential calculus an infinitesimal quantity or number is one so small that its square and all higher powers can be neglected, i.e. set to zero: we shall call such a quantity a nilsquare infinitesimal. It is to be noted that the property of being a nilsquare infinitesimal is an intrinsic property, that is, in no way dependent on comparisons with other magnitudes or numbers. An infinitesimal magnitude may be regarded as what remains after a (genuine) continuum has been subjected to an exhaustive analysis, in other words, as a continuum 'viewed in the small'. In this sense an infinitesimal may be taken to be an 'ultimate part' of a continuum: in this same sense, mathematicians have on occasion taken the 'ultimate parts' of curves to be infinitesimal straight lines.
>
> We observe that the 'coherence' of a genuine continuum entails that any of its (connected) parts is also a continuum, and accordingly, divisible. A point, on the other hand, is by its nature not divisible, and so cannot be part of a continuum. Since an infinitesimal in the sense just described is a part of the continuum from which it has been extracted, it follows that it cannot be a point: to emphasize this we shall call such infinitesimals nonpunctiform.[56]

[56]Bell, *A Primer of Infinitesimal Analysis*, 2-3.

With non-standard analysis, such infinitely small (but nonzero) real numbers were put on solid ground and were incorporated into the real number system in such a way that none of the basic features of arithmetic were undermined. In this setting, one can then define what is called *micro-continuity* (of a function f at a point a), as follows:

for all x infinitely close to a, the value $f(x)$ is infinitely close to $f(a)$.

Within smooth infinitesimal analysis, moreover, as already briefly indicated above, we get an axiomatic theory of nilsquare and nonpunctiform infinitesimals, replacing reasoning that uses the limit concept with explicit computations using infinitesimals.

As far as generality is concerned: such 'closeness' characterizations give rise to a 'spatialized' idea of generality in terms of whatever embraces or is defined over a greater *extent* of 'close' things (so that, in the limit, the most general would cover things that are extended in such a way that certain of its parts are maximally far from others).

Size

Here, continuity is basically regarded as a matter of *size*. The continuum can be said to have an unboundedly large collection of points—a proper class worth. Peirce spoke of this in terms of the continuum's *supermultitudinous* collection of points. For his part, Peirce held that there was a "transformation of quantity into quality" entailed in such a supermultitudinous collection, when points lose their individual identity and become fused together, after "enough" points were "inserted" between old points. The idea here is basically to take the notion of continuity as infinite divisibility as far as it will go—holding that the continuum is "a possibility of repeated division which can never be exhausted in any possible world, not even in a possible world in which one can complete abnumerably infinite processes."[57] In this connection, Zalamea suggests that a truly supermultitudinous continuum would be one that would not even be reachable in the cumulative cantorian set-theoretic hierarchy (by any cardinal, however big).[58]

In this context, I can also mention what Zalamea calls the "super-infinity"[59] of the class of Conway's surreal numbers, initially generated in a fashion reminiscent of Dedekind 'cuts', producing a totally ordered proper class containing the reals and the transfinite ordinals, as well as embracing infinite and infinitesimal numbers (numbers larger or smaller, respectively, in absolute value than any positive real number). Where real numbers "filled in the gaps" between the integers, through 'cuts' the surreal numbers come to fill in the gaps between Cantor's ordinal numbers. Every ordered field can be embedded in the maximal ordered field formed by the surreals. Conway calls the class of all surreals **No**. Ehrlich has argued that **No** is a unifying framework not only for the reals and ordinals but also for many non-Archimedean ordered number systems, such as those that have proven useful in connection with various non-Archimediean ordered systems, the theory of rates of growth of real functions, and non-standard analysis.[60]

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[57]Putnam, "Peirce's Continuum," in Ketner, *Peirce and Contemporary Thought*, 17. Putnam calls this Peirce's "daring metaphysical hypothesis" (ibid.).

[58]Zalamea, *Peirce's Logic of Continuity*, 14, note 19.

[59]Ibid., 42.

[60]See Ehrlich, "The Absolute Arithmetic Continuum and the Unification Of all Numbers Great and Small."

Passage

Limit of convergent sequences inherit properties of sequence

Continuity (of functions) can also be defined in terms of limits of sequences that converge. Much more generally, this approach should be related to Leibniz's particular formulation of the "law of continuity" in terms of how the "rules" characterizing the finite are "extended" to the infinite.

> In any supposed continuous transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.

In other words, for Leibniz, it is nothing more than the understanding of a general rule by means of which a sequence is generated that enables one to grasp the limit when the sequence converges. His favorite example was the succession of regular polygons progressively filling out a circle, where, even though it is not strictly true that a circle is a kind of regular polygon, and though it is "not included in any rigorous sense in the variable which [it] limits," the limit of that progression "nevertheless has the same properties as if [it] were included in the series."[61]

Local-Global Passage

Continuity as emerging via a necessary and progressive passage from the local to the global. This local-global passage perspective is realized by sheaf theory, as discussed in Chapter 6. In brief: a sheaf is not to be situated in either the local (restriction) or the global (collation) registers, but rather is to be located in the passage forged between these two,

[61] "Justification du Calcul des infinitésimals par celuy de l'Algébre ordinaire," in Leibniz, *Philosophical Papers and Letters*, 546.

in the translation system or glue that mediates between the two registers. The transit from the local to the global secured via the sheaf gluing (collatability) condition provides a deep but also precisely controllable connection between continuity (via the emerging system of transition or translation functions guaranteeing coherence or compatibility between the local sections) and generality (global sections). By separating something into parts, i.e., by specifying information locally, considering coverings of the relevant region, and enabling the decomposition or refinement of value assignments into assignments over restricted parts of the overall region (restriction condition), we are presented with a problem, a problem that in a sense can only first appear with such a "downward" movement towards greater refinement. Without having separated something into parts, we may appear to have a sort of trivial or default cohesion of parts, where, without being recognized in their separation, the parts yet remain implicit and so the glue binding them together or the rule allowing one to transit from one part to another in a controlled fashion is simply not visible. However, having decomposed or discretized something into parts, we are at once presented with this separation of parts and the problem of finding and making explicit the glue that will serve to bind them together. On this perspective, a sheaf is a way of taking information that is locally defined or assigned and decomposing those assignments in a controlled fashion into assignments over smaller regions so as to draw out the specific manner of effecting translations or gluings that obtain between those particular assignments with respect to their overlapping regions, and then using this now explicit system of gluings to build up a unique and comprehensive value assignment over the entire network of regions. In this sense, a sheaf equally involves both (i) controlled decomposition (discreteness), and (ii) the recomposition (continuity) of what is partial into an architecture that makes explicit the special form of cooperation and harmony that exists between the decomposed items, items that may have previously been detached, or which may have only *appeared* to "stick together" because we had not bothered to look closely enough.[62]

As discussed in Chapter 6, this idea of continuity in terms of passages from the local to the global creates an alignment between greater generality and the more global (and so, a corresponding alignment between the local and the less general).

Which is Better?

Frequently, disputes as to the fundamental continuity-discreteness of the universe or nature as a whole, or even to the comparative superiority of one characterization of continuity over another, boil down to some sort of claim that either continuity or discreteness is "better" than the alternative. In what follows, I outline some of the major ways these claims get developed. After doing that, I briefly discuss two of the ways it has been argued that, effectively, *neither* or *both* is "better."

Continuity is "Better"

Pragmatic

Charles Peirce held synechism to provide a foundation for his pragmaticist "research program" of fallibilism.

> I have proposed to make *synechism* mean the tendency to regard everything as continuous. The Greek word means continuity of parts brought about by surgery. [...] I carry the doctrine so far as to maintain that

[62]See Chapter 6 for more details on this.

> continuity governs the whole domain of experience in every element of it.[63]

Peirce believed that synechism could help to distinguish his pragmaticism from its usual relativistic showings. The fundamental idea of pragmaticism is contained in the maxim:

> Consider what effects, that might conceivably have practical bearings, we conceive the object of our conception to have, then, our conception of these effects is the whole of our conception of the object.[64]

According to Peirce, "the production of belief is the sole function of thought," and "belief is a rule for action, the application of which involves further doubt and further thought."[65] Moreover,

> the whole function of thought is to produce habits of action. . . To develop its meaning, we have, therefore, simply to determine what habits it produces. . . and there is no distinction of meaning so fine as to consist in anything but a possible difference in practice.[66]

In this way, the notion of pragmaticism was closely allied with *fallibilism*, the strong commitment to the idea that any one of one's current beliefs might be mistaken, that absolute certainty or infallible truths were not only unattainable but were even, in principle, undesirable. For Peirce, what characterizes the "infallibilist" is, above all, a rejection of synechism,

> because [the infallibilist] is committed to discontinuity in regard to all those things which he fancies he has exactly ascertained, and especially in regard to that part of his knowledge which he fancies he has exactly

[63] "Immortality in the Light of Synechism," Peirce, *Collected Papers of Charles Sanders Peirce.*

[64] Ibid., 5.402.

[65] "How to Make Our Ideas Clear," in Houser and Kloesel, *The Essential Peirce, Volume 1*, 127; 129.

[66] Ibid., 131.

> ascertained to be certain. For where there is continuity, the exact ascertainment of real quantities is too obviously impossible. [...] Thus scientific infallibilism draws down a veil before the eyes which prevents the evidences of continuity from being discerned.
>
> But as soon as a man is fully impressed with the fact that absolute exactitude never can be known, he naturally asks whether there are any facts to show that hard discrete exactitude really exists. That suggestion lifts the edge of that curtain and he begins to see the clear daylight shining in from behind it.
>
> [...] Once you have embraced the principle of continuity no kind of explanation of things will satisfy you except that they grew. The infallibilist naturally thinks that everything always was substantially as it is now. Laws at any rate being absolute could not grow. They either always were, or they sprang instantaneously into being by a sudden fiat like the drill of a company of soldiers. This makes the laws of nature absolutely blind and inexplicable. Their why and wherefore can't be asked. This absolutely blocks the road of inquiry. The fallibilist won't do this.[67]

This is why, according to Peirce, the only inscription that belong "upon every wall of the city of philosophy" is

> Do not block the way of inquiry.[68]

In short, the synechist wager that everything is fundamentally continuous provides a foundation for a pragmatics of knowledge that avoids all forms of dogmatism. It also appears to have moral implications, to the extent that, according to Peirce,

> Nor must any synechist say, "I am altogether myself, and not at all you." If you embrace synechism, you must abjure this metaphysics of wickedness. In the first place, your neighbors are, in a measure, yourself, and in far greater measure than, without deep studies in psychology, you would believe. Really, the selfhood you like to attribute to yourself is, for the most part, the vulgarest delusion of vanity.[69]

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[67] "Fallibilism, Continuity, and Evolution," in Peirce, *Collected Papers of Charles Sanders Peirce*.

[68] Ibid., 135.

[69] "Immortality in the Light of Synechism," ibid.

Epistemic

Towards the end of Aristotle's *Physics*,[70] Aristotle offers a brief defense of the claim that "continuous motion is *possible*," by relying on two key assumptions: (i) continuity is *better* than the alternative (mere succession), and (ii) in nature we always assume the presence of the better:

> Since there must be motion continuously, and there would be motion continuously if it is either continuous or successive, but more so if it is continuous, and it is better for it to be continuous rather than successive, and we always assume what is better to be present in nature so long as it is possible, and it is possible for it to be continuous (which will be shown later; let it be assumed now), and this can be no other motion than change of place, then necessarily change of place is primary.[71]

Moreover, in changing place, a moving thing "departs from what it is least (of all the motions)."[72] Ultimately, of the motions in place, motion in a circle is held to be "best" and even the only strictly continuous motion, since it is "more simple and complete."[73] It is simple because ultimately *indivisible* and it is complete because "the end joins up with the beginning." The ultimate justification for these two attributes appears to default to the fact that what is simple and complete is more *knowable*. Thus, the fundamental continuity of all things is not just "better" in any old sense, but "better" in that it is held to be that which ultimately guarantees the *knowability* of nature.

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[70]See *Physics*, VIII.7, and Chapter 2 of this dissertation.

[71]Aristotle, *Aristotle's Physics*, 260a21-26.

[72]Ibid., 261a22.

[73]Ibid., 265a17.

Aesthetic

We already saw how Leibniz develops his "principle of continuity" in terms of the idea that "nature never makes leaps," a law that, Leibniz elaborates, has implications not just on the whole of physics but even on a proper theory of perception and mind (*pneumatique*), through the existence of the *petite perceptions*, subject to "insensible variations." This law, he claimed,

> entails that one always passes from the small to the large and back again through what lies between, both in degrees and in parts, and that a motion never arises immediately from rest nor is it reduced to rest except through a lesser motion, just as we never manage to pass through any line or length before having passed through a shorter one.[74]

We further observed how, for Leibniz, these virtual (*virtuel*) *mikron* or imperceptible changes are one of the ways he argues against the existence of atoms, in the course of which he remarks that

> It can even be said that as a result of these tiny perceptions, the present is filled with the future and laden with the past, that everything conspires together (*sympnoia panta*, as Hippocrates said), and that eyes as piercing as those of God could read the whole sequence of the universe in the smallest of substances.
>
> [. . .] These insensible perceptions also indicate and constitute the individual, which is individuated [*caractérise*] by the traces which these perceptions preserve of its previous states, connecting it up with his present state. [. . .] It is also by means of these insensible perceptions that I explain the marvelous pre-established harmony between the soul and the body, and also between all the monads or simple substances, which takes the place of that untenable influence of the one on the others [. . .].[75]

[74] "Preface to the New Essays on the Understanding," in Leibniz, *Philosophical Essays*, 297.

[75] Ibid., 295-96.

In discussing all this in terms of pre-established harmony, something that is clearly underwritten by the hypothesis of the continuity of nature, a very important connection is established:

> It is one of the rules of my system of general harmony, *that the present is big with the future*, and that he who sees all see in that which is that which shall be. What is more, I have proved conclusively that God sees in each portion of the universe the whole universe, owing to the perfect connexion of things.[76]

But in discussing this "system of pre-established harmony" in its own right, Leibniz develops the idea that it must be such that God

> chooses rules that least restrict one another. They are also the most *productive* in proportion to the *simplicity of ways and means*. It is as if one said that a certain house was the best that could have been constructed at a certain cost. One may, indeed, reduce these two conditions, simplicity and productivity, to a single advantage, which is to produce as much perfection as is possible: thus Malebranche's system in this point amounts to the same as mine. Even if the effect were assumed to be greater, but the process less simple, I think one might say that, when all is said and done, the effect itself would be the less great, taking into account not only the final effect but also the mediate effect. For the wisest mind so acts, as far as it is possible, that the *means* are also in a sense *ends*, that is, they are desirable not only on account of what they do, but on account of what they are. The more intricate processes take up too much ground, too much space, too much place, too much time that might have been better employed.[77]

This is a repeated theme: the optimal level of *simplicity* (and "regularity" or "uniformity") *of the rules* that generate and describe the processes of nature, paired with the *greater productivity* of such rules, is what determines "the best." The idea that nature does not

[76]Leibniz, *Theodicy*, 345, §360.

[77]Ibid., 261, §208.

make leaps, then, basically serves to underwrite an aesthetic vision of nature as operating in a maximally *economical* fashion: one that always gets the most complexity, sophistication, and variety, from the simplest of materials or rules.

Discreteness is "Better"

Deflationist

This can take the form of a principled critique of the coherence of the concept of *infinitesimals*. This critique is often closely joined—and not by accident—to a critique of (certain kinds of) generals/abstract ideas, as one finds in Berkeley. Berkeley also attacked the notion that nature was infinitely divisible, on the grounds that "being is perceiving"; in an argument that should be compared to Leibniz's appeal to the existence of "petite perceptions" in his elaboration of the law of continuity, Berkley claims that

> Every particular finite extension which may possibly be an object of our thought is an idea existing only in the mind, and consequently each part thereof must be perceived. If, therefore, I cannot perceive innumerable parts in any finite extension that I consider, it is certain that they are not contained in it; but it is evident that I cannot distinguish innumerable parts in any particular line, surface or solid, which I either perceive by sense, or figure to myself in my mind: wherefore I conclude that they are not contained in it. Nothing can be plainer to me than that the extensions I have in view are no other than my own ideas; and it is no less plain that I cannot resolve any one of my ideas into an infinite number of other ideas; that is, they are not infinitely divisible.[78]

Accordingly, in setting up an isomorphism between the "simples" forming the smallest ideas into which our thought or perceptions can be resolved and the "finite extensions" that may be the objects of our thought, and in then holding that we cannot resolve any of our ideas

[78]Berkeley, *A Treatise Concerning the Principles of Human Knowledge ...*, 73.

into an infinite number of other ideas, Berkeley thought that nature should not be held to be infinitely divisible, and that the concept of infinitesimals was entirely dispensable:

> If it be said that several theorems undoubtedly true are discovered by methods in which infinitesimals are made use of, which could never have been if their existence included a contradiction in it; I answer that upon a thorough examination it will not be found that in any instance it is necessary make use of or conceive infinitesimal parts of finite lines, or even quantities less than the minimum sensible; nay, it will be evident this is never done, it being impossible. And whatever mathematicians may think of fluxions or the differential calculus and the like, a little reflexion will shew them, that in working by those methods, they do not conceive or imagine lines or surfaces less than what are perceivable to sense. They may, indeed, call these little and almost insensible quantities infinitesimals or infinitesimals of infinitesimals, if they please: but at bottom this is all, they being in truth finite, nor does the solution of problems require the supposing any other.[79]

Beyond Berkeley and such critiques of infinitesimals, another type of deflationism proceeds by treating continuity as something like the abstract negation of our experiences of finite iteration, of "going on." On these accounts, continuity is always made to collapse into discreteness, which is in turn taken to be a "fundamental fact" of human experience, and 'infinity' is held not to amount to some extension from the finite into a complete notion, but to indicate some "unlimited technique." At most, on this view, our intuitions about the infinite and our commitments to its importance reflect a questionable extrapolation from the finite mathematics of the indefinitely large.[80]

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[79]Berkeley, *A Treatise Concerning the Principles of Human Knowledge ...*, §132.

[80]Some of Wittgenstein's later "finitist" writings belong here; it is also very common among the *algorithmically* inclined. See also Lavine, *Understanding the Infinite*, for an interesting defense of the idea that 'infinite mathematics' can be reduced to a strictly finite mathematics of the indefinitely large. I note in passing that it is no accident that such approaches are nearly always *nominalistic* as well, for generals require *some* continuity condition (and most conceptions of continuity seem to inherently involve the infinite); deny, *a priori*, any robustness to the notion of continuity and it becomes very difficult, if not impossible,

Digitalists

Similar to the "deflationists," but typically putting their own *positive* theses at the forefront, some take the view that everything is discrete, that continuity ultimately is a degenerate case of some form of the discrete. This includes, most notably, the more "computational" treatments of Fredkin's "digital physics," that ultimately all quantities, including space and time, are discrete and finite; Konrad Zuse's notion that the universe is some sort of vast "computing" discrete cellular automaton; and Gregory Chaitin's "digital philosophy."[81] In general, on this view, not only is the physical universe held to be a finite (but huge) digital computer of sorts, but the true "mathematical universe" is also held to be discrete or digital; thus, for instance, even the real line is in fact "a *discrete* necklace," meaning $\mathbb{R} = h\mathbb{Z}_p$, where p is a fixed (but huge and unknowable) prime, and h is a tiny but not infinitesimal 'mesh size'.[82]

As I understand it, the wager of the "digitalists" is that we can remove continuous representations and models and not lose anything "essential" (that cannot be "simulated" by discrete counterparts): this is realized in the practical efforts to teach math to computers. If finite resource machines performing finite discrete operations can come to produce any possible continuous outputs that would be recognized as continuous (in some important sense) by human beings, these efforts would largely have succeeded in showing continuity to

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to understand connections (of whatever sort) between associated but distinct entities to be anything more than conventional, or ideal, or a matter of appearance.

[81] I ignore subtleties in the question, first raised by David Lewis, whether the analog-digital distinction might not map perfectly onto the continuous-discrete distinction.

[82] See Zeilberger, ""Real" Analysis Is a Degenerate Case of Discrete Analysis," 2-3.

be a degenerate case of the discrete. On the other hand, if it turns out that in computing some phenomenon, such as the motion of a wave, the computations cannot advance from the assumption of discrete approximations to space and time, i.e., if certain continuous processes cannot be calculated via discrete approximations, then these efforts would be thwarted. In this context, it is interesting to consider why discrete simulations of continuous phenomena—however well they may work, however simpler they may be conceptually, and despite the fact that in principle finite computations can imitate such continuous processes with as much precision as one likes—can frequently be very labor-intensive and difficult (e.g., to program) as compared to their continuous counterparts.

Computation Power

Wolfram mounts evidence for the idea that "at a fundamental level absolutely every aspect of our universe will in the end turn out to be discrete."[83] He develops this idea in terms of computation, arguing that if this were true, it would imply that there cannot be any form of continuity that violates what he calls the Principle of Computational Equivalence, where this says that systems found in the natural universe can perform computations up to a "universal" level of computational power, and that most systems do in fact attain such a maximal level of computational power (and, consequently, most systems that are not obviously simple, i.e., that pass a certain low threshold of sophistication, will be computationally equivalent). But even restricting our attention to a domain or level where a particular system

[83]Wolfram, *A New Kind of Science*, 730.

appears continuous, Wolfram asks whether at this level one could perform more sophisticated computations than in a discrete system, and says:

> My guess is that for all practical purposes one cannot. Indeed, it is my suspicion that with almost any reasonable set of assumptions even idealized perfectly continuous systems will never in fact be able to perform fundamentally more sophisticated computations.[84]

Continuous systems appear to represent a kind of detail that discrete systems do not, which is what may have suggested that they could perform more sophisticated computations. But in most examples, it does not seem that with continuous representations one goes beyond what can be generated by evolving a discrete system. Moreover, even

> To compare the general computational capabilities of continuous and discrete systems one needs to find some basic scheme for constructing inputs and decoding outputs that one can use in both types of systems. And the most obvious and practical approach is to require that this always be done by finite discrete processes.[85]

The ultimate idea, here, is thus that there will be a kind of "computational reduction" of the traditional continuous abstract models to the discrete, and since discrete systems are arguably simpler (in terms of the intelligibility of their basic ingredients and rules), we could effectively "eliminate" continuous models.

Neither

Antinomic

The idea here is to reveal some apparent incompatibility or even "contradiction" between the conclusion that the universe is continuous and the conclusion that the universe is discrete,

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[84]Wolfram, *A New Kind of Science*, 730.

[85]Ibid., 731.

by showing that there are in fact perfectly reasonable arguments for each conclusion on its own. Kant developed this antinomial approach by focusing it through the lens of the nature of composition and (in)divisibility. According to Kant, viable arguments could be made to support both of the following claims (thesis and antithesis).

Thesis: Every composite substance in the world consists of simple parts, and nothing exists anywhere except the simple or that which is composed of simples. (In other words, composites are composed of simple, indivisible parts.)[86] Kant then says

> From this it follows immediately that all things in the world are simple beings, that composition is only an external state of these beings, and that even though we can never put these elementary substances completely outside this state of combination and isolate them, reason must still think of them as the primary subjects of all composition and hence think of them prior to it as simple beings.[87]

It is not clear why the particular claim that "composition is only an external state of these [simple] beings" in fact follows; rather, it appears to follow because Kant *assumed* this in the

[86]The proof is by contradiction. The negation of the thesis should be "There exists a composite substance that does not consist of simple parts, or there exists something that is neither simple nor composed of simples." Kant begins by assuming that composite substances are not composed of simple parts. He then claims that:
If all composition "is removed in thought" (Kant, *Critique of Pure Reason*, 476, A436/B464),
then neither a composite (non-simple) part nor a simple part would remain (since there is no composition and there are no simple parts), and so "no substance would be given" (ibid.).
Therefore, either (a) it is impossible to "remove all composition in thought" or else (b) "after its removal something must be left over that subsists without any composition, i.e., the simple." In other words, either (a) one cannot imagine the absence of composition or (b) imagining the absence of composition, there must be simple (non-composite) parts. Assuming (a), that we cannot "remove all composition in thought," then *since* with regard to substances, composition is "only a contingent relation, apart from which, as beings persisting by themselves, they must persist," it apparently follows that the composite would not consist of substances. But this contradicts the assumption that we are dealing with a composite *substance*.
Therefore, (b) must be true, namely that after the "removal in thought" of composition, something must be left over that subsists without any composition, i.e., the simple." *Therefore*, every composite substance is composed of simples.

[87]Ibid., 476.

argument itself. But I do not intend to evaluate this claim. Kant simultaneously provides an argument for the antithesis.

Antithesis: No composite substance in the world consists of simple parts, and nowhere in it does there exist anything simple. (In other words, composites are composed of infinitely divisible parts.)[88]

Without having to consider whether or not these arguments lead to transcendental idealism, as Kant believed, and without displaying possible issues in the arguments themselves or evaluating the legitimacy of certain assumptions used in the arguments, suffice it to say that one could develop the "antinomial" approach in a more general fashion. The broad idea would be to somehow show that perfectly sound arguments could be made showing that the universe was continuous and also that it was discrete. Such an apparent "contradic-

[88]The proof goes like this: first assume there exists a composite substance that consists of simple parts. Then, *since* "every external relation between substances, hence every composition of them, is possible only in space," it follows that "there must exist as many parts of space as there are parts of the composite thing occupying it" (Kant, *Critique of Pure Reason*, 477). But, *assume* space is not made up of parts, and so is infinitely divisible. *Therefore*, every part of the composite must occupy a space. Therefore, in particular, the simple parts occupy space. But, *since* "everything real that occupies a space contains within itself a manifold of elements external to one another, and hence is composite, and indeed, as a real composite, it is composed not of accidents (for they cannot be external to one another apart from substance), but therefore of substances," it follows that "the simple would be a substantial composite." In other words, since every object that occupies space contains a manifold of constituents, every such object is composite. Therefore, each part, as occupying some space, is in fact a composite of substances, not simple. Contradiction. *Therefore*, no composite consists of simple parts.

This argument rests on the claim that 'everything real which occupies space' is extended, and since everything extended is divisible, and since everything is divisible is composite, everything real that occupies space is composite. Thus, the sort of composite we are considering here, in the antithesis, appears to be different (less general) than the "composite substance" being considered in the thesis. (See Grier, "Transcendental illusion and transcendental realism in Kant's second antinomy.") The meaning of "composition" in the thesis and antithesis must be different: in the thesis, composition is considered in general, whereas the antithesis considers composition of spatially extended objects. In the thesis, composition and division are the same, i.e., a division necessarily divides an entity into the elements that compose it. In the antithesis, the division produces spatial parts that constitute the composite in space. But here, one might argue that Kant is saying that space necessarily conditions any composite (taken as an appearance). And because of infinite divisibility of extension, the constitutive parts into which the object is divided are not given. So composition and division are not the same. The thesis assumes an experience of something like a substance "in itself" (which is indivisible), whereas the antithesis assumes the experience of something like space "in itself" (which is infinitely divisible).

tion" would then be used to motivate a claim to the effect that there is some unwarranted assumption (or "use of reason" itself) underlying the two claims.

Both

Dialectical

Hegel's notion of continuity and discreteness as involved in a "dialectic" can be thought of as emerging out of a critique of Kant's antinomial approach. In Hegel's hands, the two emerge as no longer problematically contradictory and reducible to the issue of the nature of "space," but rather space itself is held to be the dynamic unity through the "self-negation" of quantity (which includes, as two "moments," both the discrete and the continuous). This dialectical unity of the moments of continuity and discreteness is perhaps most fruitfully developed in terms of some of Lawvere's ideas, where dialectics is reinterpreted within the context of toposes.[89]

Conclusion

It would be far too ambitious to expect to *resolve* this issue at this moment (either in this dissertation or perhaps at the present historical moment). But, for what it is worth, I can remark that while arguments are sometimes made to the effect that perhaps "discretization should be seen as an hypothesis concerning the available technology, not the nature of the universe. It has been imposed by the technicalities, by algorithmic thinking,"[90] where such arguments typically fall back on some appeal to the greater "naturality" of the hypothesis of

[89]See Chapter 6 for details.

[90]Thom and Noël, *To Predict is Not to Explain*, 81.

continuity, it would be better, I believe, to take seriously the possibility that the "algorithmic thinking" (and, more precisely, its realization in existing technologies) is one of the most advanced ways yet of *testing* just how much of our existing models (many of which are continuous) of systems can be discretized, without losing anything fundamental. In a strange twist, by appealing to the main idea of Peirce's fallibilism (which allegedly depended on the assumption of the continuity of all things), we should remain open to the idea that, fundamentally, everything will turn out to be discretizable. Thus far at least, in spite of some of the apparent advantages of holding continuity to be "better," I see no reason or compelling argument for blocking such a possibility in principle. In the past, especially over the last century or so, many aspects of the physical world that had been assumed to be continuous have been discovered to be built up by discrete components. The idea that space itself is not a perfect continuum should also continue to be seriously considered.

In the meantime, hopefully this dissertation has contributed, in small part, to greater clarity on the many different characterizations of continuity and some of the arguments for and against maintaining the continuity or discreteness of nature; and I hope such clarity will help the reader achieve greater transparency about some of the commitments and "futures" contained in the different approaches to these matters.

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