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Exploring Total Graphs of Complete Bipartite Graphs

Dominic L J Morgan
DePaul University, dmorga19@depaul.edu

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Dominic Morgan*

Department of Mathematical Sciences

T. Kyle Petersen, PhD; Faculty Advisor

Department of Mathematical Sciences

ABSTRACT In a proper vertex coloring for a graph G, we call a vertex a b-vertex whenever this vertex is adjacent to at least one vertex of every color class. The b-chromatic number of a graph is defined to be the greatest integer k such that there exists a proper coloring on the graph with k colors where there is at least one b-vertex in every color class. This paper is on the b-chromatic number of the total graphs of complete bipartite graphs. We find bounds of the b-chromatic number for the total graphs of complete bipartite graphs and provide a method to calculate the b-chromatic number of the total graphs of complete bipartite graphs in specific conditions.

INTRODUCTION

Graph theory is the study of mathematical objects called graphs, which are defined as collections of vertices and the edges between those vertices.

Definition 1. A graph $G = (V, E)$ is a finite set V of vertices together with a set E of edges, each of which is an unordered pair of distinct vertices.

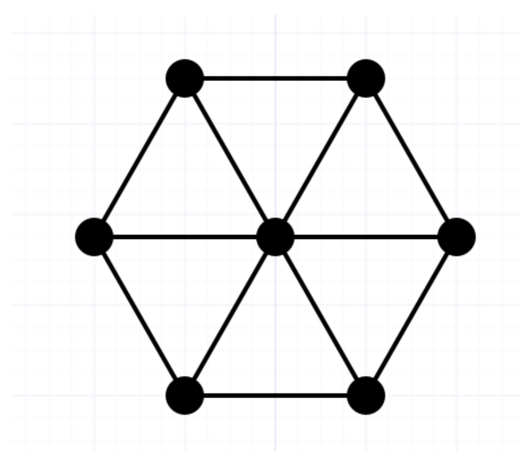

Figure 1: Depicted above is an example of a graph. In this case, the graph is the wheel graph consisting of 7 vertices, or W_7.

* *dmorga19@depaul.edu*

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Graphs come in all shapes and sizes and as such we often refer to specific families of graphs, for example: paths, cycles, wheels, trees, and bipartite graphs.

In this paper we will be investigating *bipartite graphs*.

Definition 2. A bipartite graph G is a graph whose vertices can be organized into two sets A and B, such that any edge does not have both endpoints in the same set.

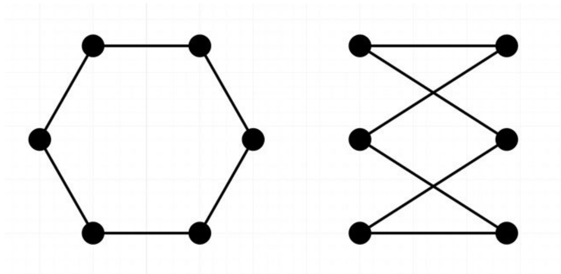

Figure 2: Depicted above is an example of two graphs that encode the same information of the same bipartite graph.

Despite looking completely different, the graphs in Figure 2 are the same graph and contain the same information but have been organized differently. In either graph we can still organize our vertices into two sets such that no two vertices in a given set are adjacent to each other. In this paper we will be concerned with *complete bipartite graphs*, which are defined as follows.

Definition 3. A complete bipartite graph, denoted $K_{a,b}$, is a bipartite graph with the property that any vertex in one of its disjoint sets A or B, is adjacent to all vertices in the other disjoint set.

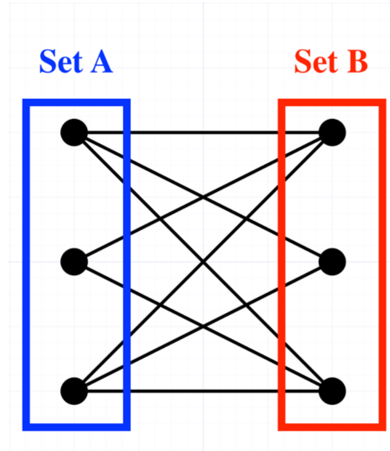

Figure 3: The above is an example of a complete bipartite graph $K_{3,3}$, with the three vertices on the left being in set A, and the right three vertices being in set B.

There is a plethora of questions one can ask about graphs and their various properties, but we will be investigating an extension of graphs called total graphs, where we construct a new graph by converting our edges into vertices and reconnecting our graph.

Definition 4: Two vertices are adjacent in the original graph if they share an edge.

Definition 5: If two edges or an edge and a vertex share an endpoint in the original graph, then it is said they are incident to each other.

Definition 6. Given a graph $G = (V, E)$, the total graph of G, denoted *T(G),* is a graph whose vertices are all vertices and edges in G. Moreover, two vertices in $T(G)$ have an edge if and only if their counterparts in G were either adjacent vertices, were an adjacent edges, or were an incident vertex and edge. That is, - $T(G) = ((V \cup E), (E \cup F \cup H))$ where $F = \{\{x, y\} \mid x \in E, y \in V, y \in x\}$, and $H = \{\{w, z\} \mid w, z \in E, v \in V, v \in w, z\}$.

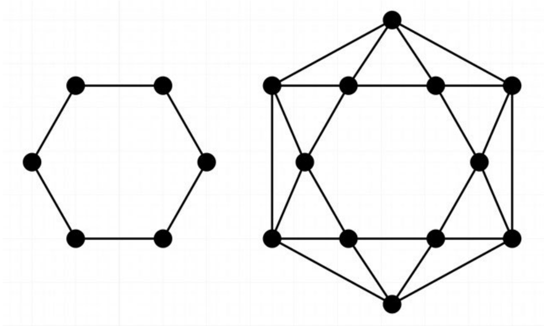

Figure 4: An example of a total graph of a bipartite graph.

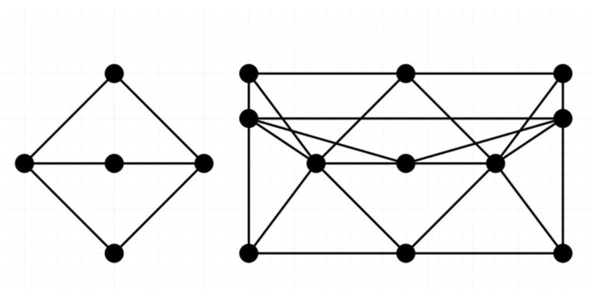

Figure 5: The complete bipartite graph $K_{3,2}$ and its total graph $T(K_{3,2})$.

Total graphs get messy. We can see from Figure 5 that our total graphs grow quickly in size and complexity for even small complete bipartite graphs. One way that we can measure this growth is the degree of the highest degree vertex, denoted $\Delta(G)$.

Definition 7. Given a graph $G=(V, E)$ and $v \in V$, we define $deg_G(v)$ to be the number of vertices adjacent to v in G.

Definition 8. $\Delta(G)$ is the degree of the highest degree vertex in G.

We see that the number of vertices a given vertex is adjacent to, or the degree of our vertex, increases in the total graph. Let G=(V, E) be a graph, then let $v \in V$. Then we know $deg_{T(G)}(v) = 2deg_G(v)$, because we know that the degree of a vertex in v is connected to deg(v) other vertices in G, by deg(v) edges. So in a total graph of G, those deg(v) edges become vertices that are adjacent to v, thus doubling deg(v). Then it follows that $\Delta(G) \leq \Delta(T(G))$.

METHODS

Graph Coloring

The fundamental process of this paper is graph coloring, which consists of assigning colors to vertices of a graph. Of course, to produce anything interesting we impose certain restrictions to our coloring process. One such restriction that we require our coloring to be *proper*.

Definition 9. A proper coloring of a graph $G = (V, E)$ is a function f : $V \rightarrow \mathbb{N}$ such that for $v_1, v_2 \in V$ and $e = \{v_1, v_2\} \in E$ that $f(v_1) \neq f(v_2)$.

We will interpret each value as a color, hence our terminology for "coloring". As discussed previously, writing out and coloring total graphs would be difficult, so instead we can utilize a *proper total coloring*.

Definition 10. A proper total coloring of a graph $G = (V, E)$ is a function f : V ∪ E ⟶ ℕ that takes vertices and edges in G and assigns them each a color such that no pair of adjacent vertices, no pair of adjacent edges, and no pair consisting of an incident edge and vertex are assigned the same color.

Any total proper coloring of a graph is a proper coloring of the total graph, and vice versa. Because we are only concerned with graph colorings of $T(G)$, we can use a proper total coloring to minimize the legwork of writing out and then coloring our total graphs. This works because our colored edges act the same as vertices in $T(G)$ and therefore obey the same rules for our coloring purposes.

It is natural to ask, "What is the smallest number of colors one can use for a proper coloring?".

Definition 11. The chromatic number of a graph G is the fewest number of colors required for a proper coloring and is notated $\chi(G)$.

Take for example the path graph P_3 in Figure 6. We can pick a vertex and color it color 1, then we can move onto another vertex adjacent to it. Then we must use another color, color 2, to maintain a proper coloring. Finally, we see that our last vertex can be colored with color 1. Then we see that we have $\chi(P_3) = 2$.

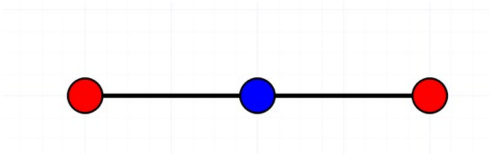

Figure 6: Depicted above is the path graph P_3 being colored with 2 colors.

It is natural to explore questions related to the minimum number of colors we can use to color a graph, as well as the maximum number of colors. These are all questions that have real world applications that warrant investigation. In fact, within the last decade graph coloring has been used to improve the postal distribution system [2]. In addition, graph coloring also has applications in systems used to plan airplane layovers and healthcare data analysis [1].

B-Chromatic Number

The notion of b-colorings was introduced by Irving and Manlove in 1999 [3] and is expounded up by Jakovac and Peterin [4].

Definition 12. Given a properly colored graph G, a b-vertex is a vertex that is adjacent to at least one other vertex of every other color.

Definition 13. A b-coloring is a proper coloring of a properly colored graph G such that for each color class, there is at least one b-vertex of that color.

Definition 14. A total b-coloring is a proper total coloring of a graph G such that for each color class, there is at least one b-vertex of that color.

Definition 15. The b-chromatic number of a graph, denoted $\varphi(G)$, is the maximum number of colors you can use in a b-coloring.

Our chromatic number is necessarily smaller than our b-chromatic number. If we had a smaller b-chromatic number than our chromatic number that would be a contradiction because then we have found a coloring using less colors than our chromatic number. We cannot have more colors than the degree of the vertex with the most adjacent vertices. Then it follows that $\chi(G) \leq \varphi(G) \leq \Delta(G) + 1$ [4].

We can give a better upper bound for our b-chromatic number. If our b-chromatic number is n, then by definition we need at least n vertices with degree at least $n - 1$. We define this new bound as our m-degree.

Definition 16. The m-degree of a graph G, written $m(G)$, is the maximum integer n such that G has at least n vertices of at least degree $n - 1$.

Then it follows that $\varphi(G) \leq m(G)$, as the b-chromatic number will either reach the m-degree or will fall short.

Putting this all together we can check the cycle C_3. We can see that our m-degree is 3, as we have 3 vertices with degree 2. Then we can verify that C_3 has a b-chromatic number that is equal to its m-degree and $\varphi(C_3) = 3$. In this case we have $\chi(C_3) = \varphi(C_3) = m(C_3)$.

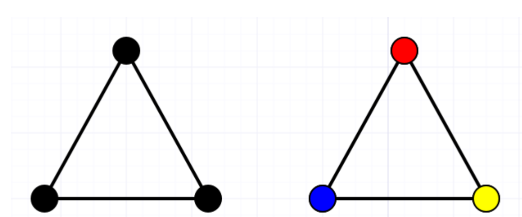

Figure 7: Pictured above the cycle graph C_3 colored with three colors.

It is important to note that we do not always reach the upper bound of the b-chromatic number set by the m-degree, as we will see in later examples of $K_{a,b}$.

RESULTS

B-Chromatic Number of Complete Bipartite Graphs

Complete bipartite graphs have some curious properties when it comes to b-coloring. Despite having relatively high m-degrees, complete bipartite graphs only ever reach a b-chromatic number of 2.

Lemma 1. The b-chromatic number of the complete bipartite graph $K_{a,b}$ is 2. That is, $\varphi(K_{a,b}) = 2$.

Proof: Let $G = K_{a,b}$, with the vertices in G organized into two disjoint sets A and B, with cardinalities a and b respectively. If we color every vertex in our first set A with color 1, and all vertices in B with color 2, then we have a b-coloring of G with 2 colors, as every vertex of color 1 that is in set A is connected to a vertex in B of color 2 and the same with the other vertices in B. Hence, $\varphi(K_{a,b}) \leq 2$.

Now we will show we cannot have $\varphi(K_{a,b}) \geq 3$. Assume that $\varphi(K_{a,b})=k \geq 3$, for some $k \in \mathbb{N}$. Vertices will either be in a set A or a set B, with vertices within the same set not being adjacent to each other. Without loss of generality let $v \in$ A be a b-vertex of color 1, which means that there are vertices $v_2, v_3 \in B$ of colors 2 and 3, respectively. But then, we see that we cannot have b-vertices for both colors 2 and 3 in either set A or set B, as no vertices in set A can be colored with colors 2 and 3, as then we would have an improper coloring. Thus, we see that we can have at most b-vertices of one color in each set and we cannot have $\varphi(K_{a,b}) \geq 3$.

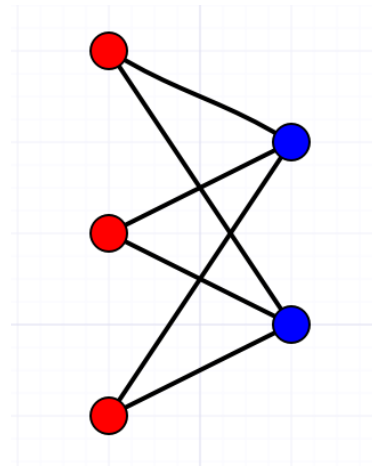

Figure 8: Depicted above is complete bipartite showing $\varphi(K_{a,b}) = 2$.

As mentioned previously we can use our m-degree to act as an upper bound for our b-chromatic numbers for both $K_{a,b}$ as well as $T(K_{a,b})$.

Lemma 2. The m-degree of $K_{a,b}$ is equal to $min(a, b)$+1.

Proof. Without loss of generality, we can say $b \leq a$. Then we know that there will be b vertices in our set B and a vertices in our set A. Then our vertices in set A will have degree b, and our vertices in set B will have degree a. Then because $b \leq a$, we know that our highest degree vertices will be our vertices in set B, but we only have b many of them. So then for us to have an m-degree of $a + 1$ we would need $a + 1$ vertices of degree at least a, which we do not have. On the other hand, we can have an m-degree of $b + 1$ as we have a vertices of degree b as well as b vertices of degree at least b. Similarly, if we let $a \leq b$, we find $a + 1$ to be our m-degree. Finally, we cannot have an m-degree of any of the values between b and a because either our vertices will either have degree a or degree b. If we want an m-degree for values between a and b than the vertices with degree a will contribute, but then we run into the same problem where the vertices of b will not. Hence, we can find the m-degree by taking $min(a, b)+1$.

Lemma 3. $m(T(K_{a,b})) = a + b + 1$.

Proof. In $T(K_{a,b})$ we have a vertices connected to b vertices by b edges, so they have degree $2b$ in $T(G)$. Then we also have b vertices connected to a vertices by a edges, so they have degree $2a$ in $T(G)$. Then for our edge vertices we have ab edge vertices with 2 incident vertices and $a + b - 1$ adjacent edges. Thus, we have ab edge vertices with degree $a + b + 1$ in $T(G)$. Then the highest degree we have are the b vertices with degree $2a$, but we do not have $2a + 1$ vertices of degree $2a$, so we cannot have an m-degree of $2a$. But then we have at least $a + b + 1$ vertices of degree at least $a + b$, because we have b vertices with degree $2a$ as well as the edge vertices with degree $a + b + 1$. Then we have $ab + b \geq a + b + 1$. So, we have an m-degree of $a + b + 1$.

Unlike in regular complete bipartites, the upper bound provided by the m-degree proves to be helpful in figuring out $T(K_{a,b})$. It is clear that the graphs generated by taking the total graph of a complete bipartite graph will not be bipartite, and so it is not unreasonable to expect that the total graph will reach the m-degree, as these total graphs will not be restricted by the bipartite condition. However, we now show that at least in the case where we have $T(K_{n,2})$ we cannot reach our m-degree of $n + 2 + 1$, but we can reach $n + 2$. We can prove this by defining an algorithm that provides us with a correct total b-coloring with $n + 2$ colors.

Theorem 1. $\varphi(T(K_{n,2})) = n + 2$.

Proof. We will proceed via contradiction, that is, suppose that $\varphi(T(K_{n,2})) = n + 3$. Let $G = K_{2,n}$ for some natural number n. Then let $\{x_1, x_2\}$ be set B and $\{y_1, y_2, \ldots, y_n\}$ be set A. We will perform a total coloring on G with $n + 3$ colors. Observe that necessarily we must have $\lfloor \frac{n}{2} \rfloor$ edges be both b-vertices and adjacent to one another. This is because n vertices in set A of $K_{n,2}$ only have degree 2, and cannot be b-vertices. Without loss of generality, we can let $\{x_1, y_1\}$ be a b-vertex of color 1, then we can make it a b-vertex by coloring its neighboring vertices colors $\{2, \ldots, n\}$. Observe that the low-degree end y_1 and $\{x_2, y_1\}$ are colored 2 separate colors. Then color another edge adjacent to $\{x_1, y_1\}$ and incident to x_1. Without loss of generality call this edge $\{x_1, y_2\}$, then on the low degree end we have y_2 and $\{x_2, y_2\}$ then to maintain a proper coloring we see that y_2 and $\{x_2, y_2\}$ must have opposite colors of y_1 and $\{x_2, y_1\}$. Then for our third adjacent edge $\{x_1, y_3\}$, we see that y_3 and $\{x_2, y_3\}$ cannot be either of the necessary colors, so we cannot make this edge a b-vertex. Thus, we cannot have $\varphi(T(K_{n,2})) \neq n + 3$.

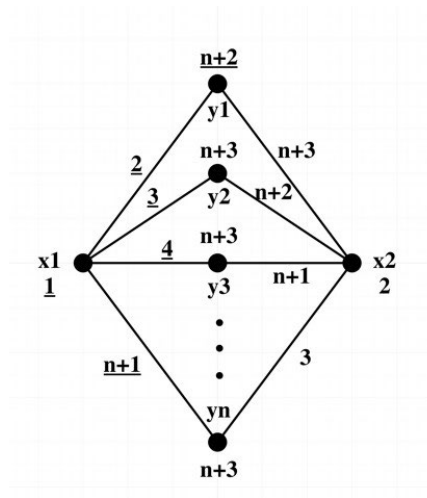

Figure 9: Illustration of algorithm showing $\varphi(T(K_{n,2})) \neq n + 3$. Underlined elements are b-vertices. Ellipses indicate all $n - 4$ vertices and their respective edges.

Now we show that we can have $\varphi(T(K_{n,2})) = n + 2$. Let $G = K_{2,n}$ for some natural number n. Then we will perform a total b-coloring on G with colors $\{1, 2, 3, \ldots, n+2\}$. We will make both $\{x_1, x_2\}$ be color 1, then $\{y_2, \ldots, y_n\}$ be color $n + 2$, and y_1 be color $n + 1$. Then color the edges between x_1 and y_i for $1 \leq i \leq n$ with colors $\{2, \ldots, n + 1\}$.

Then all the vertices $\{x_1\}$, $\{x_1, y_2\}$, $\{x_1, y_3\}$, …,$\{x_1, y_n\}$ are b-vertices. Now we just need to make a b-vertex with color n+2, so we color the edge $\{x_2, y_1\}$ with color n+2 and all the edges adjacent to itself with the colors $\{2, …, n\}$. Observe that $\{x_2, y_1\}$ is adjacent to all other colors and is therefore a b-vertex. Thus, we have $\varphi(K_{n,2}) = n + 2$.

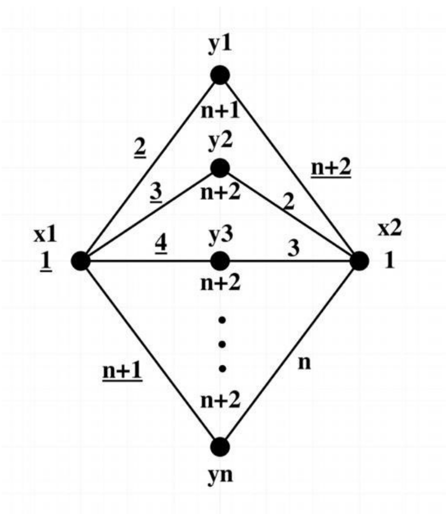

Figure 10: Illustration of the coloring algorithm in the proof. Underlined elements are b-vertices, and ellipses represent all *n* – 4 vertices and their respective edges.

Indeed, finding a coloring algorithm is a useful method of proving facts about the b-chromatic number of the total graphs of complete bipartite graphs. However, in these proofs the algorithm benefits greatly from the fact that we have 2 vertices in one set of vertices in our complete bipartite graph. Beyond 2 vertices, it becomes harder to find these patterns and build a method of coloring these graphs. For select cases we can show that $\varphi(T(K_{a,b})) = a+b$. For example, we can define another coloring algorithm to show that $\varphi(T(K_{4,3})) = 7$.

Theorem 2. $\varphi(T(K_{4,3})) = 7$.

Proof. We will proceed via contradiction, that is we will assume $\varphi(T(K_{4,3})) = 8$. We need at least 5 of our edges to be b-vertices, as none of our vertices in set *A* can be b-vertices, but all 3 vertices in set *B* can be possible b-vertices. Then without loss of generality we can choose one edge vertex and color its neighbors to make it a b-vertex. This edge has an endpoint of degree 4- the high degree endpoint. Then because we have 5 edges that have to be b-vertices and only 4 vertices on the outside- as depicted in figure 11- at least two of these edges must be incident to one another. Then let another edge incident to our initial edge be a b-vertex. Then we have 3 neighbors left with 3 colors to use. Those 3 colors are the same color as our initial edge vertex used at its low degree end. Then we know that these neighbors form a cycle of 4, so then we have 2 vertices, 4 edges, and 2 more high degree vertices. The low degree vertices and incident edges must be some permutation of these 3 colors. So, if you pick an edge in this cycle, it is already connected to the other 2 colors and on the high degree end, so it must share one of those two other colors and because we need every adjacent vertex to a different color, it cannot be a b-vertex, so we lose 4 edges and two high degree vertices. Then we would need to repeat this process again, which would mean we would not be able to color this graph with 8 colors.

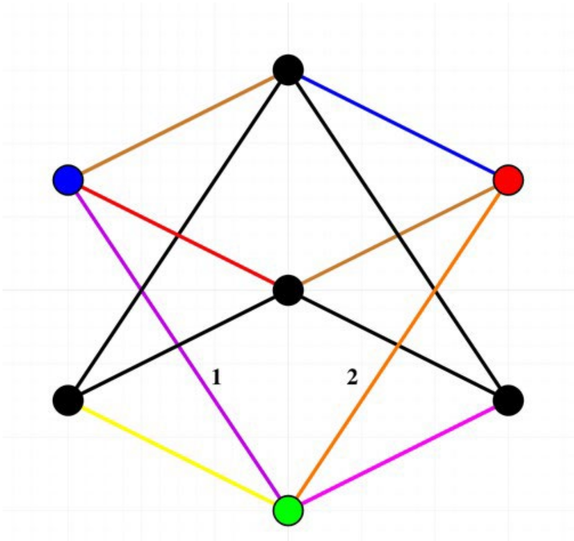

Figure 11: Depicted above is a visualization of the proof $\varphi(K_{4,3}) \neq 8$. Black edges and vertices are uncolored. The violet and orange edges are b-vertices 1 and 2.

Then to show $\varphi(K_{4,3}) = 7$, it suffices to show a valid total b-coloring with 7 colors, which is provided by Figure 12, pictured below.

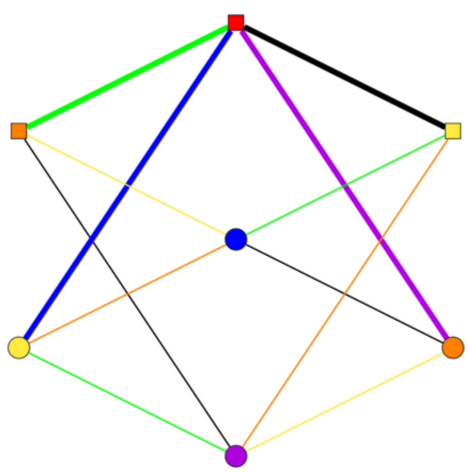

Figure 12: B-coloring of $K_{4,3}$ using 7 colors. Bold edges and square vertices are b-vertices.

DISCUSSION

Conclusion

In this paper we have discussed b-chromatic numbers for complete bipartite graphs and the b-chromatic number for $T(K_{n,2})$, as well as calculating the b-chromatic number for a select case $T(K_{4,3})$.

Finding the b-chromatic number total graphs for complete bipartite graphs is a difficult process. The graphs grow quickly and often do not seem to meet their m-degree. Usually, it is trivial to color $\varphi(K_{a,b}) = a + b - 1$, but is harder to show that $\varphi(T(K_{a,b})) = a + b$ or $\varphi(T(K_{a,b})) = a + b + 1$.

Conjecture: $\varphi(T(K_{a,b})) = a + b$.

In the future we hope to explore the other cases for $\varphi(T(K_{a,b}))$, namely when a = b and a ≥ b. It is our conjecture that $\varphi(T(K_{a,b})) = a + b$. We have been able to establish this for small cases and lack a good, generalized algorithm for proving the other more complicated cases. Computer assistance could prove to be useful in verifying this conjecture for larger cases.

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