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Daniel Gawel
DePaul University, dgawel1@depaul.edu

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Increasing the *b*-Chromatic Number of Complete Bipartites by Deleting Edges

Daniel Gawel*

Department of Mathematical Sciences

Emily Barnard, PhD; Faculty Advisor

Department of Mathematical Sciences

ABSTRACT A *b*-coloring of a graph, $G=\langle V,E\rangle$, is a proper coloring in which each color has at least one vertex that is adjacent to every other color. The *b*-chromatic number for *G*, denoted $\varphi(G)$, is the largest number of colors you can use to make a *b*-coloring. For most graphs, $\varphi(G)$, is around the *m*-index, an upper bound for $\varphi(G)$ based on the graph's degree sequence. However, for the complete bipartite graph $K_{m,n}$, $\varphi(K_{m,n})=2$ even though the *m*-index can be arbitrarily large. In this paper we will delete edges from complete bipartites so we can increase their *b*-chromatic number. Specifically, we will delete the fewest amount of edges that it takes to maximize the *b*-chromatic number of complete bipartites.

INTRODUCTION

This paper is all about graphs and *b*-colorings. Graph Theory is a subfield in mathematics that deals with mathematical structures known as graphs. Graphs appear all over computer science through neural networks and GPS frameworks. Coloring these graphs with certain constraints has had various applications in the past with map coloring and task scheduling. In this paper we will focus on coloring a specific type of graph known as complete bipartites. Let's introduce some terms first.

Definition 1. A **Graph** is a set of vertices, *V*, and a set of edges, *E*. If two vertices are connected by an edge then they are **adjacent** to each other. The **degree** of some vertex, *v*, denoted, *deg(v)*, is the number of vertices adjacent to it.

Example 1. Let Figure 1 be a graph called G_e. Let $G_e=\langle V,E\rangle$ where: $V=\{1,2,3,4,5,6,7\}$ and $E=\{\{1,2\},\{2,3\},\{2,6\},\{3,4\},\{4,6\},\{4,7\},\{4,5\}\}$. Let Table 1 be a list of degrees for G_e's vertices.

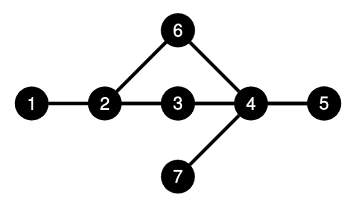

Figure 1. The Graph G_e.

v	1	2	3	4	5	6	7
$deg(v)$	1	3	2	4	1	2	1

Table 1. Degree Sequence of G_e.

* dgawel1@*depaul.edu*

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Definition 2. A b-coloring of a graph, $G=\langle V,E\rangle$, happens when we color the graph with two constraints:

1. No two adjacent vertices share the same color (such a graph is known as a proper coloring).
2. Each color has at least one vertex that is adjacent to at least one vertex of every other color (such vertices are known as ***b*-vertices**).

The concept of a b-colorings was first introduced by Irving and Malone in 1999. If you color a graph with the least amount of colors you can use to properly color it, then it will also be a b-coloring. Therefore, every graph has a b-coloring (see [3], section 2).

Example 2. Let Figure 2 be a b-coloring of G_e containing two colors, red and green. Each of those colors contains a vertex that is adjacent to every other color. In this case every vertex is a b-vertex.

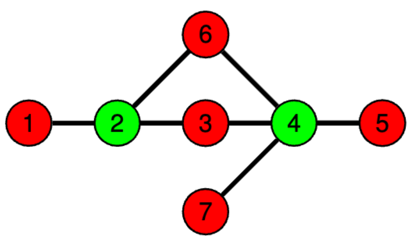

Figure 2. A b-coloring of G_e

Definition 3. The b-chromatic number for some graph G, denoted $\varphi(G)$, is the maximum number of colors we can use to b-color G.

Example 3. Let Figure 3 be a maximal b-coloring of G_e. Notice, the graph now contains three colors. In this case, vertices 2, 3, 4, and 6 are all b-vertices.

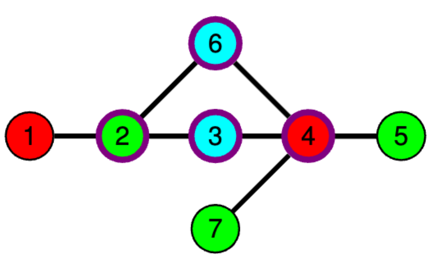

Figure 3. A maximal b-coloring of G_e.

Notice, if $\varphi(G)=n$ for some graph, then G must have at least n vertices with degrees of at least n-1. Using this idea, we can find an upper bound for $\varphi(G)$.

Definition 4. The m-index of a graph G, denoted $m(G)$, is an upper bound for $\varphi(G)$ based on the degree sequence of G. That is, $\varphi(G) \leq m(G)$.

Example 4. Let us go back to Table 1 and calculate the m-index for G_e. First ask does G_e have 2 vertices with degrees greater than 1? Yes! Move onto the next value. Does G_e have 3 vertices with degrees greater than 2? Yes! Does G_e have 4 vertices with degrees greater than 3? No! Therefore, $m(G_e)=3$, meaning $\varphi(G_e) \leq 3$. In this case, $\varphi(G_e)=m(G_e)$.

For most graphs, $\varphi(G)$ is around $m(G)$, meaning $\varphi(G)=m(G)$ or $\varphi(G)=m(G)$-1. However, there are exceptions!

Definition 5. A Complete Bipartite Graph $K_{m,n}$, is a graph whose vertices can be split up into two subsets V_1 and V_2 with $|V_1|=m$ and $|V_2|=n$ so that no two vertices in the same subset are connected and each vertex in V_1 is adjacent to each vertex in V_2.

Example 5. Let Figure 4 be the complete bipartite $K_{5,4}$. Notice, even though $m(K_{5,4})=5$, $\varphi(K_{5,4})=2$.

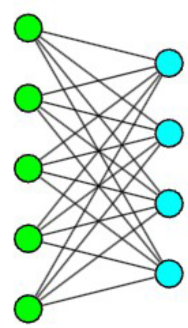

Figure 4. The graph $K_{5,4}$ with $\varphi(K_{5,4})=2$

In general, for every complete bipartite graph, $\varphi(K_{m,n})=2$ no matter how large $m(K_{m,n})$ is. Having $m(G)$ and $\varphi(G)$ so far a part is strange.

In this paper we will delete edges from such complete bipartites and increase their *b*-chromatic number in the fewest edges possible.

When it comes to graphs and researching their *b*-chromatic numbers, there is more discourse centered around deleting vertices from graphs compared to solely deleting edges from graphs. R. Balakrishnan and S. Francis Raj found an upper bound for any vertex deleted graph that is connected and has more than 5 vertices (see [2]).

Even then, research centered around edge deletion from complete bipartite graphs are more about bounds of the resulting graph's *b*-chromatic number. Most notably, P. Francis and S. Francis Raj discovered an upper bound for $\varphi(G\text{-}e)$ based on $\varphi(G)$, where e is a single edge. (see [1], Theorem 6)

We will delete multiple edges from complete bipartites and find their *exact* *b*-chromatic number.

METHODS

To obtain our results we have written an algorithm using SageMath, meant to delete various combinations of edges from graphs and have it output the *b*-chromatic number of the resulting graph.

In short, this is how the algorithm worked:

1. Take in a Graph, $G=\langle V,E\rangle$.
2. Get all combinations of edges.
3. For *each* combination, delete the edges from the graph and spit out the *b*-chromatic number.

Clicking this link will direct you to the code with various examples of complete bipartites as inputs alongside their outputs.

RESULTS

Let us restate the question: $\varphi(K_{m,n})=2$ no matter how large $m(K_{m,n})$ is. Can we increase $\varphi(K_{m,n})$ by deleting edges?

Before we continue we will introduce a new term that is very important in understanding the results.

Definition 6. A matching is a set of edges with no common vertices.

Example 6. Let $M=\{\{1,2\},\{3,6\},\{4,5\}\}$ and $N=\{\{1,2\},\{1,4\},\{5,6\}\}$ be sets of edges for some graph. Since all the vertices in M are unique, then the edge set M is a matching. However, since the vertex 1 appears in N twice, then the edge set N is *not* a matching.

Using our algorithm, there is one pattern that stood out: If the set of edges deleted from any complete bipartite was a *maximal* matching, meaning we delete the maximum amount of edges which don't share the same endpoint, then the *b*-chromatic number would jump from $\varphi(K_{m,n})=2$ to $\varphi(K_{m,n}\text{-}E)=n+1$, where $n<m$ and E is the set of edges of edges we delete from $K_{m,n}$.

Theorem 1. *Suppose we have $K_{m,n}$ with $n<m$. Then:*

1. *$\varphi(K_{m,n}\text{-}E) \leq n+1$, where E is any set of edges you delete from $K_{m,n}$.*
2. *The least amount of edges that it takes to achieve $\varphi(K_{m,n}\text{-}E)=n=1$ is $|E|=n$, where E is a maximal matching.*

We will explain this theorem through an example.

Example 7. Suppose we have $K_{4,3}$. We will delete a maximal matching, E', from $K_{4,3}$, to increase $\varphi(K_{4,3})=2$ to $\varphi(K_{4,3})=4$. Let the maximal matching we delete from $K_{4,3}$ be $E'=\{\{1,5\},\{2,6\},\{3,7\}\}$. Figure 5 is just the graph $K_{4,3}$. Figures 6, 7, and 8 each represent deleting one of the edges from the maximal matching.

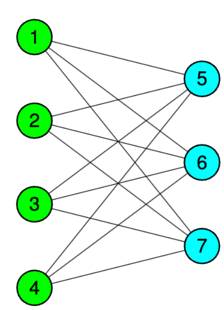

Figure 5. $\varphi(K_{4,3})=2$.

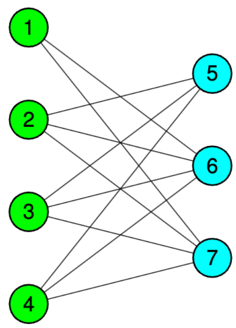

Figure 6. $\varphi(K_{4,3}\text{-}\{\{1,5\}\})=2$.

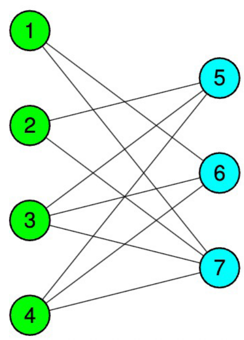

Figure 7. $\varphi(K_{4,3}\text{-}\{\{1,5\},\{2,6\}\})=2$.

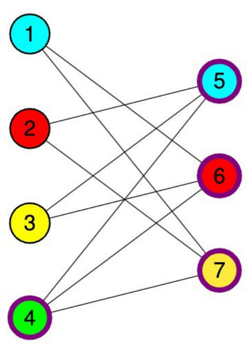

Figure 8. $\varphi(K_{4,3}\text{-}\{\{1,5\},\{2,6\},\{3,7\}\})=4$.

As you can see, the edges in $\{\{1,5\},\{2,6\}, \{3,7\}\}$ share no common vertex, thus making it a matching. If we would try to add another edge to this set, then it would no longer be a matching since we would have to repeat a vertex. Thus, this edge set is a maximal matching of $K_{4,3}$.

Notice, vertices 5, 6, and 7 which all belong to the right side (side with fewer vertices) become *b*-vertices with each one being a different color. Vertex 4 on the left side also becomes a *b*-vertex as we didn't delete any edges from it.

Once a maximal matching was deleted from $K_{4,3}$ then we achieved a *b*-chromatic number of $\varphi(K_{4,3}\text{-}E')=4$. Notice, $m(K_{4,3})=4$ so we cannot get a higher *b*-chromatic number by deleting any more edges. Therefore, we achieved the maximal *b*-chromatic number for $\varphi(K_{4,3}\text{-}E)$ where E is *any* set of edges deleted from $K_{4,3}$.

In general, $m(K_{m,n})=n+1$. Deleting edges from any graph only has the effect of decreasing its *m*-index, as you lower the degree sequence. Therefore, for any set of edges you delete from $K_{m,n}$, E, $m(K_{m,n}\text{-}E) \leq n+1$, where $n<m$. Therefore $\varphi(K_{m,n}\text{-}E)\leq n+1$.

If we deleted an edge, for example $\{1,5\}$, then both 1 and 5 are *able* to become the same color now, because two adjacent vertices cannot be the same color in a *b*-coloring, and when you delete an edge between two vertices they are no longer adjacent. Even though we delete the edge $\{1,5\}$ in Figure 6, vertices 1 and 5 didn't change into the same color until we deleted the last edge in the maximal matching in Figure 8. These vertices cannot change to the same color before then because neither one can become *b*-vertices without making the graph an improper coloring, meaning two vertices of the same color would be adjacent, breaking constraint 1 of a *b*-coloring. Thus, you need to delete *at least* 3 edges from the maximal matching before the *b*-chromatic number increases. Therefore, we know that the least amount of edges it takes to reach this maximal *b*-chromatic number is 3, where the edges we delete is a maximal matching.

DISCUSSION

So what did we find? Using this idea of a matching we can efficiently increase the *b*-chromatic number of a complete bipartite. Can we apply this idea of a matching to other graphs?

Definition 7. The Complete Graph K_n is the graph of n vertices where every vertex is adjacent to every other vertex. Note, $\varphi(K_n)=n$.

Example 8. Let Figure 9 be the complete graph K_6. Notice, each vertex in K_6 is a b-vertex.

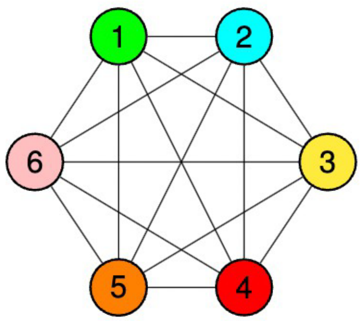

Figure 9. $\varphi(K_6)$=6.

If you look carefully, K_6 is a complete 6-partite graph, meaning you can break it up into 6 subsets, with vertices in the same subset not being adjacent to each other, but they are adjacent to every other vertex in the other subsets. In this case we have 6 subsets each with 1 vertex in them. We can rewrite K_6 as $K_{1,1,1,1,1,1}$. The 6 numbers represent the 6 subsets of vertices, and the numbers themselves represent how many vertices are in those subsets, similar to how we write out complete bipartites. In general, K_n is a complete n-partite graph.

Let's delete a matching, {{1,2},{3,4},{5,6}}, from this graph and see what happens. Figures 10, 11, and 12 are the results of deleting these edges.

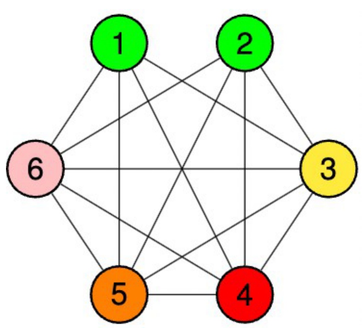

Figure 10. $\varphi(K_6$-{{1,2}})=5.

Notice, the b-chromatic number of the graph decreased from 6 to 5. Now the graph is a complete 5-partite, which can be written as $K_{2,1,1,1,1}$.

Now let's delete a second edge form the graph and examine what happens to the b-chromatic number.

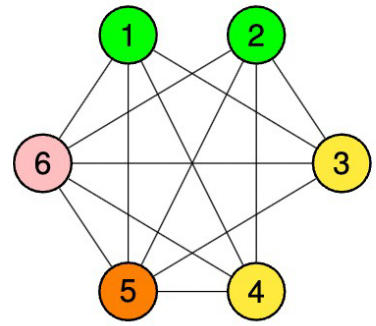

Figure 11. $\varphi(K_6$-{{1,2},{3,4}})=4.

Now we deleted a matching with 2 edges and the b-chromatic number of the graph decreased from 5 to 4, changing the graph from a complete 5-partite to a complete 4-partite. The graph can now be written as $K_{2,2,1,1}$.

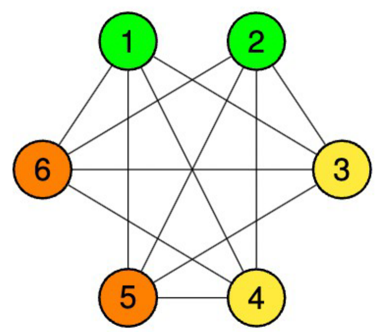

Figure 12. $\varphi(K_6$-{{1,2},{3,4},{5,6}}=3.

As we deleted a matching with 3 edges, the b-chromatic number of the graph decreased from 4 to 3. Figure 12 is now a complete 3-partite, which can be written as $K_{2,2,2}$. At this point, we ended up deleting a maximal matching. However, notice how each time we deleted an edge, the b-chromatic number decreased by 1. This leads us into our concluding theorem.

Theorem 2. *If E is a matching, then* $\varphi(K_n\text{-}E)=n\text{-}|E|$.

As we saw, each edge you delete from a matching, lowers the b-chromatic number of a complete graph.

In this case, our matching had 3 edges, and the complete 6-partite graph changed to a complete 3-partite. This changed the b-chromatic number from 6 to 3. Similar to the case of complete bipartites, if we deleted the edge {1,2}, then vertices 1 and 2 both become the same color, since they are no longer adjacent to each other. However, they are still adjacent to every other vertex in the graph. Since every other vertex is also adjacent to each other, this lets the color change happen right after the edge is deleted.

Unlike complete bipartites, deleting a matching from complete graphs decreases its *b*-chromatic number. This is due to the inherent structure of both graphs. Complete bipartites are split up into two subsets of vertices which restricts the graph from getting close to its *m*-index, whereas a complete graph will always hit its *m*-index.

To conclude, this idea of a matching is a powerful tool when dealing with *b*-chromatic numbers and complete *n*-partite graphs, like complete bipartites or complete graphs. If we want to efficiently increase the *b*-chromatic number for any complete bipartite, then deleting a maximal matching is essential.

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