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b-Chromatic Number of the Graph Power of a Star Graph

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ABSTRACT In this paper we focus on the newly introduced *b*-colorings of a graph *G.* A *b*-coloring is a proper coloring such that for each color class, there exists at least one vertex that is adjacent to every other color. The *b*-chromatic number of a graph *G* is the largest number *k* such that *G* admits a *b*-coloring with *k* colors. This paper will introduce the *b*-chromatic number of some interesting graphs. Several operations of graphs are defined, and the *b*-chromatic number of those operations are found. All graphs in this paper are simple, connected, non-regular graphs. In our main result we compute the *b*-chromatic number of the graph power of a star.

INTRODUCTION

This paper is interested in a section of graph theory related to *b*-colorings. All graphs, *G*, in this paper are finite, simple, undirected, non-trivial, and connected. *V(G)* represents the set of vertices of *G,* and *E(G)* the edges. A finite graph is defined as a graph *G* such that $|V(G)| = n$ for some positive integer *n*. A simple graph is one that contains no edges connecting a vertex to itself. An undirected graph is a graph that contains no direction from vertex to vertex. A non-trivial graph is a graph *G* such that $|V(G)|$ >0. Finally, a connected graph is a graph *G;* there always exists a path containing every vertex. A path in a graph *G* is a sequence of vertices such that one vertex is adjacent to to the next vertex via an edge.

Given a graph *G,* a proper coloring of *G* assigns a color to every vertex $v \in V(G)$ such that no adjacent vertices share the same color (Figure 1). All colors exist in the color set *C*. It is common to represent colors as integers such that *C = {1, 2, 3, 4, ..., k}* for some integer *k* colors. You can separate the set of vertices of *G* into color classes. If vertices share the same color, they exist in the same color class. In this paper, all colorings are proper colorings.

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Figure 1. A proper coloring of P_4.

Coloring graph problems were introduced by mathematicians and computer scientists in 1852. An intuitive question that might come up right away is: if you are given any graph *G,* what is the fewest number of colors that can completely color *G*? The smallest number of colors in any coloring of a graph *G* is called the *chromatic number* of *G* and is denoted by $\chi(G)$. The *b- chromatic number,* $\varphi(G)$, which can be intuitively used to denote the largest number of colors used to color a graph such that each color class contains a vertex that is adjacent to all other colors, was introduced by Irving and Manlove in 1997 [5], [Definition 1]. A problem that was proposed by famous mathematician Augustus De Morgan is the Four-Color Theorem, which stood open for over 150 years. Augustus De Morgan conjectured that every map can be colored in at most four colors. At the time, detailed maps of the world were being created. Countries like the United States were divided up into many small parts, and coloring such maps created a problem for cartographers – a problem which needed a graph theory solution (Figure 2).

To see how we model this problem with graph theory, we let each territory be a vertex, and we connected a pair of vertices if and only if the corresponding territories are adjacent to each other. Each such graph of territories is planar, meaning that no pairs of edges intersect. Adjacent territories cannot be the same color because then the territories would be indistinguishable. Therefore, a proper coloring is needed to make each territory distinct. A lot of effort has been put into finding the chromatic number of any simple planar graph, and the conclusion is that given a planar graph G, $\chi(G) \leq 4$. For more information on the Four-Color Theorem, mappings, combinatorial hypermaps, and what has been done after the four-color theorem; consider the following. [4]

Some other work has been done in postal mail sorting systems and *b*-colorings. Many sorting systems rely on density of address blocks to determine hierarchical necessity. The method of finding the *b*-chromatic number of a graph *G* consequently identifies those dense areas of *G.* A *b*-coloring with *k* colors contains *k* vertices that connect to all other color classes. The *b*-chromatic number, which is of interest to postal mail systems, finds the maximum number of colors that can be used in a graph such that there exists a vertex in each class that is connected to every other color class. For example, suppose we are looking at a certain area of downtown Chicago, where each address is a vertex. Vertices are adjacent to each other if one address has received mail from another address in the past 6 months. In this graph there may be some extremely dense vertices like schools, restaurants, or government buildings. But finding the *b*-vertices of this graph shows the most relevant addresses in the area and therefore sets a precedent for future hierarchical postal sorting improvisations. For a more detailed explanation consider the following. [3]

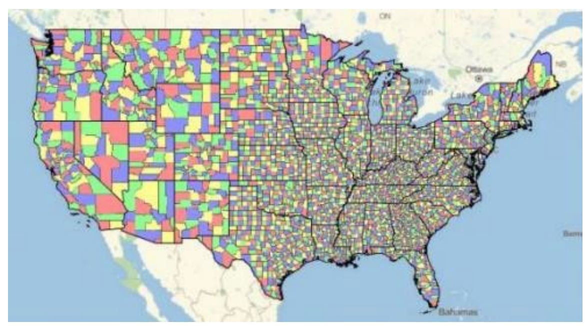


Figure 2. A proper coloring of the counties of the United States of America using four colors.

Power graphs have been useful in the field of computer science and are well studied. The results in this paper aim to explore non-regular power graphs. For information on simple power graphs and regular power graphs consider [2],[6]. The p-th power graph, denoted G^p , is a graph obtained from *G* by connecting vertices at distances *p* or less. More formally,

$$V(G^p) = V(G) \text{ and } E(G^p) = \{(u, v) : d(u, v) \leq p\}: u, v \in G.$$

The function *d(u,v)* represents the distance between the vertices *u* and *v*. The distance between u and v is the shortest path of edges from u to v. This is not to be confused with the diameter of a graph G, denoted as Diam(G), the diameter of a graph represents the longest distance between two points $u,v \in G$. The diameter is a common general bound as it limits the size of a given graph. Other bounds on a graph may include the vertex set, but sometimes we would like a vertex set to be unbounded for proof purposes. The Distance k-graph of G, denoted G_k has the vertex set of G and the edge set $\{(u, v) : d(u, v) = k\}$. It is clear then that $G_k \subseteq G^k$. Graphs of interest for distance k-graphs are the hyper- cube graph Q_n, the folded cube graph F_n, halved graphs, and distance regular graphs. Many subtopics have been extensively studied, such as trees, regular graphs, cubic graphs, and cartesian products. [6][2][5].

Fact 1. For any graph G of order n, if Diam(G) ≤ *p, then* $\varphi(G^p) = n$, *with* $p \geq 2$.

Proof. Since Diam$(G) \leq p$, it is trivial to see that G^p is a complete graph, so $\varphi(G^p) = n$.

Several notations will be used for describing graphs and sub-graphs. Notations P_n, C_n, K_n, S_n, $K_{m,n}$ stand for the path, cycle, complete, star, and complete bipartite graphs respectively. A path graph is a visual representation of the path described in the introduction. A cycle graph is a path graph such that the first vertex in the sequence is the same as the last vertex. A complete graph is a graph such that all vertices are adjacent to each other. A star graph contains one central vertex, commonly denoted v_0, and n vertices adjacent to v_0. A bipartite graph is a graph that can be split into two subsets $A, B \subseteq V(G)$: for all elements $a \in A$ and $b \in B$ there does not exist an element such that $a = b$. A complete graph is the bipartite graph as described just previously, however every vertex in A is adjacent to every vertex in B. Any graph with a chromatic number of two is considered bipartite. The star graph S_n can be written as $K_{1,n}$ (Table 1). For more information consider the following.[1] [8]. These graphs are areas of interest for computer scientists when dealing with power graphs. Bounds on the b-chromatic number have been found for the power graph of cycle graphs, path graphs, and complete graphs. For more information on power graphs and the b-chromatic number, consider [2].

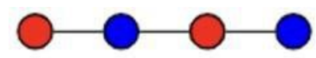
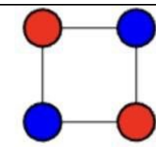
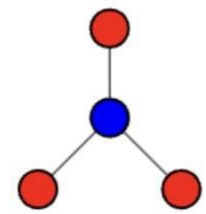
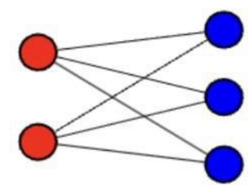
The path graph P_4 on four vertices	
The cycle graph C_4 on four vertices	
The star graph S_3 on three outer vertices.	
The complete bipartite graph $K_{2,3}$ with 2 vertices in one set and 3 vertices in the other.	

Table 1. A proper coloring of $P4$, $C4$, $S3$, $K2,3$.

METHODS

In math research, we discover our main results by using precise definitions, using inequalities to provide bounds for our computations, and computing small examples. We will include all three of these elements in this section, beginning with a precise definition of a b-coloring.

Let $C = \{c_1, c_2, c_3 \ldots c_k\}$ where k is the number of desired colors in a graph. We can refer to a graph coloring as a function $f : V(G) \rightarrow C$. It is common for each element in C to be an integer for simplicity, however the set may include colors for visual references. In this paper, we will use integers to represent the colors so therefore the function $f : V(G) \rightarrow \mathbb{Z}_+$. The $\mathbb{Z}_+$ here simply means the set of all positive integers. If this function is still unclear, then imagine assigning every vertex in a graph a positive integer. You may use the same color multiple times; however, the same vertex may not have two different colors. This is the definition of a function.

Definition 1. Let G be a graph with a proper coloring. A b-vertex is a vertex that is adjacent to every other color. A b-coloring is a proper coloring when there exists at least one b-vertex of each color (Figure 4).

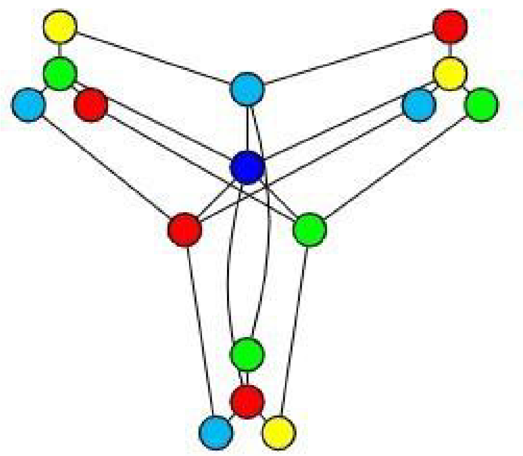


Figure 4. A b-coloring of a graph G.

Definition 2. The b-chromatic number, denoted $\varphi(G)$, is the largest positive integer k such that there exists a b-coloring with k colors.

Intuitively, the b-chromatic number is the maximum number of colors such that there exists a vertex in every color class that is adjacent to at least one vertex of every other color class. The clear relation to the adjacency matrix $A(G)$ and the adjacency hierarchy leads us to the definition of the m-degree.

Definition 3. $m(G) := max\{1 \leq i \leq n : d(x_i) \leq i - 1\}$ where $n = |V(G)|$.

In other words, the m-degree of a graph G is the maximum integer i such that there exist at least i vertices of at least degree $i - 1$. By this definition, $\varphi(G) \leq m(G)$. The elements x_i, that satisfy the inequality $d(x_i) \leq i - 1$, are called dense vertices. In other works, they may be called b-vertices or dominating-vertices. A vertex v with $deg(v)-1 = k$ can, but does not have to, be a b-vertex. Refer to the definition of a b-vertex and a b-coloring [Definition 1]. The lower bound of the b-chromatic number can be either the largest complete sub-graph K_n in G known as the clique number, denoted $\omega(G){:}|\omega(G)| = n;$ or the chromatic number $\chi(G)$. The chromatic number is always greater or equal to the clique number, therefore the chromatic number can be used as a better general lower bound. For most graphs, the b-chromatic number will either be equal to $m(G)$ or $m(G) - 1$. Note that just because a graph can reach a certain m-degree does not therefore mean the graph can be colored with that number. The m-degree is simply an upper-bound. First, we will prove that a graph has a certain upper-bound, then we will prove whether it can reach it or if it cannot.

Corollary 1. $\omega(G) \leq \chi(G) \leq \varphi(G) \leq m(G)$

Definition 4. $\Delta(G) = max\{deg(v_i) : v_i \in G\}$ or the highest degree of a vertex in a graph G.

Additionally, $\delta(G) = min\{deg(v_i) : v_i \in G\}$, or the smallest degree of a vertex in a graph G. Areas of interest are degree-regular graphs where $\delta(G) = \Delta(G)$ and therefore the m-degree, $m(G) = deg(v) + 1$. This makes finding the b-chromatic number of regular graphs simple, as the upper-bound is $\varphi(G) \leq deg(v) + 1$ and most regular graphs will reach this bound. [5][6]

Definition 5. The power graph of G, denoted as G^p, has a vertex set $V(G)$ and an edge set $E(G^p) = \{(u, v) : d(u, v) \leq p\}$.

One of the general operations for any graph G that is part of a graph's family, is its power graph. This function is not one-to-one, and its inverse does not exist for all graphs, as such, finding the root of G^p is considered NP-Hard.

RESULTS

Lemma 1. $G^1 = G$

Proof. Let $p=1$. By the definition of a power graph, we must have an edge between any vertex with distance less than or equal to 1. It is therefore trivial to see that connecting all vertices with distance 1 in G does not change the graph and therefore G^1 is isomorphic to G.

Theorem 1.

$$\varphi(S_n^k) = \begin{cases} 2, & if\ k = 1 \\ n+1, & if\ k \geq 1 \end{cases}$$

Proof. The kth-power graph of G connects $v \in V(G)$ if the distance between two vertices is at most k. Additionally, $\forall u, v \in S_n$, $d(u, v) \leq 2$, as you can always travel from u to the center vertex v_0 and then to v. $(S_n)^2$ is the graph of S_n such that vertices distances at most 2 are connected by an edge. It is trivial to see that every vertex in S_n is connected to every other vertex. Visually, if v_o is moved to the outside to create a cycle, the graph is isomorphic to K_{n+1}, therefore its b-chromatic number is the same as K_{n+1}. The b-chromatic number of K_{n+1} is $n+1$, therefore $\varphi(S_n^k)=n+1$ for $k > 1$.

Additionally, this can be verified through Fact 1.

DISCUSSION

Subtopics discussed during this research project have been the cartesian product of certain families of graphs. We looked at the star graph, the line graph of the star graph, the total graph of the star graph, the power graph of the star graph, and caterpillar graphs [1][9]. The cartesian product gives insight on a possible operation between graphs. There are very few well-studied operations between graphs, a few of which are the rooted product, the lexicographic product, the strong product, the tensor product, and the cartesian product [7]. The most used product between graphs is the cartesian product and in our research, we found that b-chromatic numbers for the cartesian product of degree regular graphs are easier to find than non-regular graphs. Next, we will briefly define the cartesian product and for the following Lemma we will define a n-regular graph.

Definition 6. The Cartesian product of two graphs G and H is denoted $G\square H$. $V(G\square H) = V(G) \times V(H)$. If there exist vertices $u_1, v_1 \in G \wedge u_2, v_2 \in H$, then there exists an edge between (u_1, u_2) and (v_1, v_2) if, and only if, $u_1 = v_1\ and\ u_2 \sim v_2$ or $u_2 = v_2\ and\ u_1 \sim v_1$.

Defintion 7. A regular graph is a graph $G: \forall v \in V(G), d(v) = k$ for some positive integer k. Consequently, the graph is considered k-regular.

Lemma 2. Given a graph G that is d_1-regular and a graph H that is d_2-regular, the cartesian product is $(d_1 + d_2)$-regular.

Proof. A short proof using the definition of the cartesian product of two sets shows that for any fixed vertex $u \in G$, u must have d_1 adjacent vertices and for any fixed vertex $v \in H$, v must have d_2 adjacent vertices. Therefore, by the definition of the cartesian product, any fixed vertex $(u, v) \in V(G\square H)$ must also be adjacent to all d_1 vertices that were originally adjacent to u and must also be adjacent to all d_2 vertices that were originally adjacent to v. Because of this, $deg((u, v)) = d_1 + d_2$ and if every vertex has an identical degree, then it must be true that $G\square H$ *is* $(d_1 + d_2)$-regular.

Corollary 2. $\varphi(G\square H) \leq d_1 + d_2 + 1$.

Although regular graphs are important in defining recurring similar data structures, like cyclic organic molecules or set arrays in multiple dimensions, there exist more applicable uses of graph colorings in *non*-regular graphs. For example, given a non-regular graph P_n and regular graph C_k, finding the b-chromatic number of the cartesian product of these two graphs is non-trivial. In addition to looking at regular graphs, we looked at the cartesian product of two star graphs $K_{1,n}$. In [6], it was found that $\varphi(K_{1,n}\square K_{1,n}) = n + 2$, *if* $n \geq 2$. However, we believe we have generalized the result to find that,

Conjecture 1. $\varphi(K_{1,n}\square K_{1,m}) = m + 2$, *if* $n \geq m$.

Conclusion

The research project has left some questions unanswered that we wish to leave as open problems to the public.

Conjecture 2. We strongly believe that the *b*-chromatic number of the tensor product of two stars, $\varphi(S_n \times S_m) = m + n + 1$.

This has not been verified for all *m,n.*

Question 1. What is the relation between the total graph function and the Cartesian product of a graph with itself? Is the *b*-chromatic number of the cartesian product $G\square G$ always greater than or equal to total graph $T(G)$?

Question 2. The power graph of a cycle C_4 and the power graph of a star S_3 are isomorphic. This becomes problematic when trying to take the inverse of the power graph because the function is not one-to-one. If a graph does not contain the sub-graphs C_4 and S_3 is there a way to define a "root graph" function that when given G^p will return G? In other words, is it possible to construct a function $\sqrt[p]{G}$?

Question 3. A star graph S_n is isomorphic to the complete bipartite graph $K_{1,n}$. If we consider Conjecture 1 as a base case, is it possible to find $\varphi(K_{u,n}\square K_{v,m})$ using mathematical induction?

Question 4. General bounds for the *b*-chromatic number of the cartesian product of graphs with girth ≥ 5 have been considered, however bounds for girths ≤ 4 have not been generalized and are considered NP-hard as the girth gets too small to consider all possibilities.

Question 5. Given that you can find the power graph of any graph using the theorem provided in this paper and previous lemmas for cycles and paths, is it possible to subdivide any graph into ranked trees based on distance? What does this tell us about the graph's chromatic number, *b*-chromatic number, and dense vertices? For any graph *G*, if it contains a high amount *n* of dense vertices at a large enough distance away from each other (determined by the power graph), are there bounds on the *b*-chromatic number based on *n*?

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