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Mapping Polynomial Dynamics

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ABSTRACT We explore the complex dynamics of a family of polynomials defined on the complex plane by $f(z) = az^m(1+z/d)^d$, where a is a complex number not equal to zero, and $m, d \geq 2$. These functions have three finite critical points, one of which has behavior that differs as we change our parameter values. We analyze the dynamical behavior at this critical point, with a particular interest in the structures that appear in the filled Julia set $K(f)$ and the basin of infinity $A_\infty(f)$. The behavior of the family is extremely sensitive to our inputs for a, m and d. We examine the connectedness of the filled Julia set, determine a region that is contained in the basin of infinity and a region contained in the component of the filled Julia set containing zero. Then we use these regions to find regions in the parameter space where the filled Julia set is disconnected or connected, respectively.

INTRODUCTION

We explore the family of functions $\mathcal{F}$ where every $f \in \mathcal{F}$ has the form,

(1) $$f_{a,m,d}(z) = az^m\left(1+\frac{z}{d}\right)^d.$$

The parameter a is a complex number not equal to zero and $m, d \geq 2$. We focus on $m \geq 2$ because the $m = 0$ case was studied by Bodelón *et al.* in [2] and the $m = 1$ case was studied by Fagella in [3]. All $f \in \mathcal{F}$ are defined on the complex plane, $\mathbb{C}$. For more background information on complex dynamics, see Beardon in [1].

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METHOD

We first introduce the notion of iteration. When iterating a function, we pick a starting point z_0, apply the function repeatedly, and analyze the behavior of the iterates. Iterating a function at a point z_0 entails finding $f(z_0)$, then plugging this output back into the function to find $f(f(z_0))$, and repeating this process n times for n iterates. If we were concerned with the first, $n = 1$, iterate, we would write $z_1 = f^1(z_0)$, and for $n = 2$, $z_2 = f^2(z_0) = f(f(z_0))$. Hence we denote the nth iterate of z_0 recursively, so if $z_n = f^n(z_0)$, then $z_{n+1} = f(f^n(z_0))$. For example, let $g(z) = z^2$ and consider the case where $z_0 = 2$. The first two iterates are $g^1(2) = g(2) = 4$, and $g^2(2) = g(g(2)) = g(4) = 16$.

While the individual iterates can be interesting on their own, we are more concerned with their behavior as we let $n \to \infty$. In some cases, as $n \to \infty$, the iterates are unbounded. For example, the iterates are unbounded at $z_0 = 2$ for $g(z)$ as defined above. We define our iterates as unbounded when for all $R > 0$, there exists an integer N such that $|f^n_{a,m,d}(z)| > R$ for all $n > N$. When z has an unbounded orbit through iteration, we write $|f^n_{a,m,d}(z)| \to \infty$ as $n \to \infty$. However, this isn't always the case. Again considering $g(z)$, when $z_0 = 0$ or $z_0 = 1$, the iterates converge to a fixed point. A fixed point, z_f, satisfies the following relationship: $f(z_f) = z_f$. If one iterate is a fixed point, by definition every iterate after it evaluates to that point. More concretely, if $f^m(z_0) = z_0$, then for all $n \geq m$ $f^n(z_0) = z_0$.

We refer to the sequence of values $f^n(z_0)$ with $n = 1, 2, ...$ as the *orbit* of z_0. At each $z_0 \in \mathbb{C}$, the orbit is bounded or unbounded. Bounded orbits include convergence to fixed points and periodic cycles. With periodic cycles, the iterates alternate through a finite set of distinct values. This polarization allows us to break the complex plane into components—a component with bounded orbits and a component with unbounded orbits. We now formally define these components.

Definition 1. The *filled Julia set*, $K(f)$, for a polynomial map f is the set of all $z \in \mathbb{C}$ that have a bounded orbit under f:

$$K(f) = \{z \in \mathbb{C} \mid f^n(z) \not\to \infty \ as \ n \to \infty\}.$$

Definition 2. The *basin of infinity*, $A_\infty(f)$, is the complement of $K(f)$:

$$A_\infty(f) = \{z \in \mathbb{C} \mid f^n(z) \to \infty \ as \ n \to \infty\}.$$

Hence we can pick any $z \in \mathbb{C}$ and follow its behavior as n gets big to see if it belongs to the filled Julia set or the basin of infinity. Figure 1 includes both a filled Julia set, the black region, and basin of infinity, the colored region, from our family of functions. Values in the black region have bounded orbits, while values in colored regions have unbounded orbits. Different colors indicate how fast z escapes to infinity. We use the program FractalStream to draw our Julia sets and parameter spaces.

FIGURE 1. The filled Julia set, $K(f)$, and the basin of infinity, $A_\infty(f)$, of $f_{-.2+.48i,2,3}(z)$. The filled Julia set is the black region, and the basin of infinity is the colored region.

To begin our analysis, we identify critical points for our family of functions, as they are essential to our understanding of polynomial dynamics. In our results, we use critical points to generate our parameter spaces and discuss connectedness of $K(f)$. A point $z \in \mathbb{C}$ is a finite *critical point* for a polynomial f if $f'(z) = 0$. We find critical points for $m \geq 2$ and $d \geq 2$ by equating the first derivative of $f_{a,m,d}(z)$ to 0 and solving for z. So we have

$$f'_{a,m,d}(z) =$$
$$az^{m-1}\left(1 + \frac{z}{d}\right)^{d-1}\left[z + m\left(1 + \frac{z}{d}\right)\right],$$

which is equal to zero when $z \in \{-d, 0, \frac{-md}{d+m}\}$. At $z = -d$, the function maps to zero for all $a \in \mathbb{C} \setminus \{0\}$. Zero is not only one of our other critical points, but also a fixed point, as $f_{a,m,d}(0) = 0$ for all a. So both $-d$ and 0 always get mapped to zero, making their orbits bounded under $f_{a,m,d}$. The behavior at our last finite critical point, $z = \frac{-md}{d+m}$, is more complicated and will be studied in subsequent sections.

RESULTS

Connectedness. We define a *connected domain* as a domain that is not disconnected. A compact domain D on the complex plane is defined as *disconnected* if and only if there exists a Jordan curve γ which separates D. A Jordan curve is a closed loop that does not intersect itself. With a connected domain, it is impossible to draw a Jordan curve that separates D. Examples of connected and disconnected domains and filled Julia sets are included in Figure 2.

The top panels in Figure 2 include (a) connected and (b) disconnected domains. Figure 2(b) shows an example of a Jordan curve γ_1 that separates the domain into components, thereby indicating that the domain is in fact disconnected. Note that the connected region in (a) is connected because there is no such Jordan curve that isolates distinct components of the domain.

Figure 2's bottom panels include (c) connected and (d) disconnected filled Julia sets of $f_{a,2,3}$. In Figure 2(c), the colored region is the basin of infinity, $A_\infty(f)$ and the black region is the filled Julia set, $K(f)$. Note that $A_\infty(f)$ is connected in both (c) and (d), but $K(f)$ is only connected in (c) because in (d) a Jordan curve γ_2 can isolate distinct components.

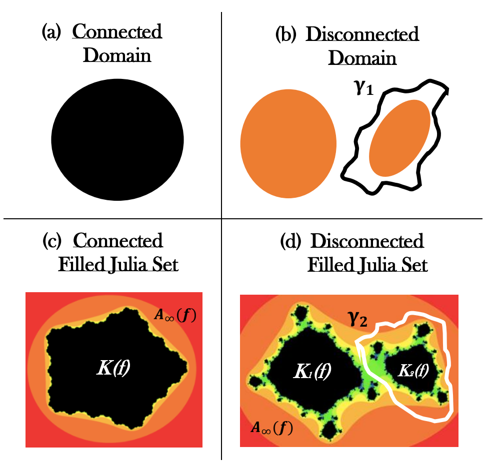

FIGURE 2. (a) A connected domain (b) A disconnected domain with Jordan curve γ_1 (c) A connected filled Julia set $K(f)$ of $f_{-.2+.48i,2,3}(z)$ (d) A disconnected filled $K(f)$ of $f_{-1.3-1.3i,2,3}(z)$ with Jordan curve γ_2

As one can see from Figure 2, for some a values $K(f)$ is connected. A connected $K(f)$ is included in Figure 2(c). For other a values, $K(f)$ is disconnected. A disconnected $K(f)$ is included in Figure 2(d). We are particularly interested in identifying which a yield these behaviors, and we do this by way of the following theorem:

Theorem 1. (Theorem 9.5.1 in [1])
Let f be a polynomial with
$\deg(f) \geq 2$. Then the following are equivalent:

(i) $K(f)$ is connected;

(ii) f has no finite critical points in $A_\infty(f)$.

Now that we have some more insight on the necessary conditions for a connected $K(f)$, we take advantage of Theorem 3.1 in our proof of Proposition 1 given below.

Proposition 1. *Consider the free critical point $c_f = \frac{-md}{m+d}$ and choose any a in the parameter space.*

(i) For all a where $|f^n_{a,m,d}(c_f)| \to \infty$ as $n \to \infty$ $K(f)$, is disconnected.

(ii) For all a where $|f^n_{a,m,d}(c_f)| \not\to \infty$ as $n \to \infty$, $K(f)$ is connected.

Proof. Our family of functions has three finite critical points: $c = 0, -d, \frac{-md}{m+d}$. Zero is a fixed point and $-d$ always maps to zero, so neither of these critical points will be in $A_\infty(f)$. The last finite fixed point $c_f = \frac{-md}{m+d}$, however, could either go to infinity or have a bounded orbit.

(i) First consider the case of an unbounded orbit, so choose a, m and d such that

$|f^n_{a,m,d}(c_f = \frac{-md}{m+d})| \to \infty$ as $n \to \infty$. Then by definition $c_f = \frac{-md}{m+d}$ is contained in $A_\infty(f)$, so there is a finite critical point of f in $A_\infty(f)$. By the contrapositive of Theorem 3.1, it follows that $K(f)$ is disconnected.

(ii) Now consider the case of a bounded orbit. Suppose the free critical point is not in $A_\infty(f)$. This means $|f^n_{a,m,d}(c_f = \frac{-md}{m+d})| \not\to \infty$ as $n \to \infty$. When this is the case, there are in fact no finite critical points in $A_\infty(f)$ and by Theorem 3.1, $K(f)$ is connected. □

An example of a parameter space is included in Figure 3. When we generate parameter spaces, we fix m, d and $z = \frac{-md}{m+d}$ and vary a in Equation 1. For a in black regions of the parameter space, the free critical point has a bounded orbit. When a is in a colored region, the orbit of the free critical point is unbounded, with the color indicating how fast it escapes.

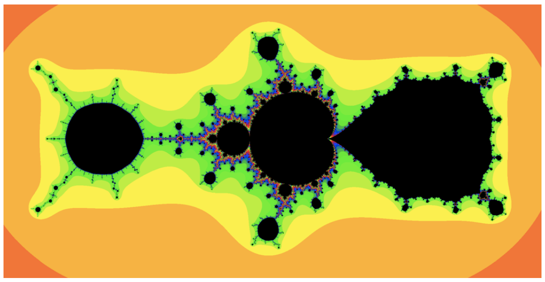

FIGURE 3. The parameter space for $m = 2, d = 3$.

With Proposition 1, we can take any a in the parameter space and know if $K(f)$ is connected or not. Looking back at Figure 2, we now know by Proposition 1 that the a value in (c) came from a black region of the parameter space, and the a value in (d) came from a colored region of the parameter space.

Now that we have studied the connectedness of the filled Julia sets, we investigate bounds on $K(f)$.

Escape Radius. In this subsection, we show the filled Julia set is bounded. Also, all z not in $K(f)$ are in $A_\infty(f)$. With this in mind, we find a specific radius in the dynamical plane past which every z is contained in $A_\infty(f)$. We define this radius below.

Definition 3. We define r_∞ as an *escape radius* if when $|z| > r_\infty$, then $|f^n_{a,m,d}(z)| \to \infty$ as $n \to \infty$.

In order to analyze our escape radius, we first introduce the following Lemma.

Lemma 4. *Fix $r > 0$ and $K > 1$. Suppose $|f(z)| > K|z|$ for all $|z| > r$. Then $|f^n(z)| \to \infty$ as $n \to \infty$.*

Proof. Let $r > 0$ and $K > 1$. Suppose $|f(z)| > K|z|$ for all $|z| > r$. Note that as $|z| > r$ and $K > 1$, we can write $|f(z)| > K|z| > Kr > r$, or

$$|f(z)| > r. \tag{2}$$

We begin with induction to prove $|f^n(z)| > K^n|z|$ for all $n \geq 1$ and all $|z| > r$.

The base case for $n = 1$ is true by assumption. Now assume $|f^n(z)| > K^n|z|$ for some $n \geq 1$ and for all $|z| > r$. Consider the following inequality:

$$\begin{aligned} |f^{n+1}(z)| &= |f^n(f(z))| \\ &> K^n|f(z)| \text{ by the induction} \\ &\quad \text{hypothesis and Equation 2} \\ &> K^nK|z| \\ &\quad \text{by the base case} \\ &= K^{n+1}|z|. \end{aligned}$$

Hence, $|f^{n+1}(z)| > K^{n+1}|z|$ holds for $|z| > r$ and it follows from induction that $|f^n(z)| > K^n|z|$ for all $n \geq 1$ and $|z| > r$. Note that as $n \to \infty$, $K^n \to \infty$. Thus, by the Squeeze Theorem $|f^n(z)| \to \infty$ as $n \to \infty$. □

We propose the following escape radius for our family of functions $\mathcal{F}$.

Proposition 2. *Let $r_\infty = \max\{2d, 2\left(\frac{d^{d+m}}{|a|}\right)^{1/(m-1)}\}$. Then for $|z| > r_\infty$, $|f^n_{a,m,d}(z)| \to \infty$ as $n \to \infty$.*

Proof. Set $r_\infty = \max\{2d, 2\left(\frac{d^{d+m}}{|a|}\right)^{1/(m-1)}\}$. For $|z| > d$ and $m \geq 2$, we have

$$|f_{a,m,d}(z)| = |a||z|^m \left|1 + \frac{z}{d}\right|^d$$

$$\geq |z|^{m-1}\left(\frac{|z|}{d} - 1\right)^d |a||z|.$$

Since $|z| \in \mathbb{R}^+$, we can define the following real-valued function where $x = |z|$:

$$h(x) = x^{m-1}\left(\frac{x}{d} - 1\right)^d.$$

We need $h(x)$ to be positive and strictly increasing for $x > d$. First note that $h(d) = 0$. Now we examine the first derivative of $h(x)$ to show that $h(x)$ is in fact increasing:

$$h'(x) = x^{m-1}d\left(\frac{x}{d}-1\right)^{d-1}\left(\frac{1}{d}\right)$$
$$+(m-1)x^{m-2}\left(\frac{x}{d}-1\right)^{d}$$
$$= x^{m-2}\left(\frac{x}{d}-1\right)^{d-1}\left[x+(m-1)\left(\frac{x}{d}-1\right)\right],$$

which is positive for $x > d$ and $m \geq 2$. Hence, $h(d) = 0$ and $h'(x) > 0$ so $h(x)$ is positive and strictly increasing for $x > d$.

We now break into cases based on the size of $|a|$:

(i) Suppose $|a| \leq d^d$. Our escape radius is defined as a maximum of two terms, so it must be at least as large as both values. We therefore have $r_\infty \geq 2\left(\frac{d^{d+m}}{|a|}\right)^{1/(m-1)}$ and $r_\infty \geq 2d$. Since $|a| \leq d^d$, we have $\frac{1}{|a|} \geq \frac{1}{d^d}$, and we can write $r_\infty \geq 2\left(\frac{d^{d+m}}{|a|}\right)^{1/(m-1)} \geq 2\left(\frac{d^{d+m}}{d^d}\right)^{1/(m-1)} = 2(d^{m/(m-1)}) > 2d > d$ (since $m \geq 2$ and $d \geq 2$).

Now if $|z| > r_\infty > 2d > d$, we have $h(|z|) > h(r_\infty)$ (as $h(x)$ is positive and strictly increasing for $x > d$.

$$|f_{a,m,d}(z)| \geq h(|z|)|a||z|$$
$$> 2^{m-1}\left(\frac{d^{d+m}}{|a|}\right)\left(\frac{r_\infty}{d}-1\right)^{d}|a||z|$$
$$= 2^{m-1}d^m(r_\infty - d)^d|z| > |z|.$$

So $h(|z|)|a| > 1$ as $h(|z|)$ strictly increases, so $|f^n_{a,m,d}(z)| \to \infty$ as $n \to \infty$ by Lemma 4.

(ii) Suppose $|a| > 1$. If $|z| > r_\infty \geq 2d > d$, then

$$|f_{a,m,d}(z)| \geq h(|z|)|a||z|$$
$$> (2d)^{m-1}|a||z| > |z|.$$

So $h(x)|a| > 1$ and $|f^n_{a,m,d}(z)| \to \infty$ as $n \to \infty$ by Lemma 4. ☐

Figure 4 includes our escape radius, r_∞, when $a = 7.03$, $m = 3$, and $d = 2$. By Proposition 2, our escape radius is given by $r_\infty = \max\{2d, 2\left(\frac{d^{d+m}}{|a|}\right)^{1/(m-1)}\} = 4.28$. The filled Julia set is entirely contained in the disk defined by this escape radius, with every $z \in \mathbb{C}$ larger than r_∞ escaping to infinity.

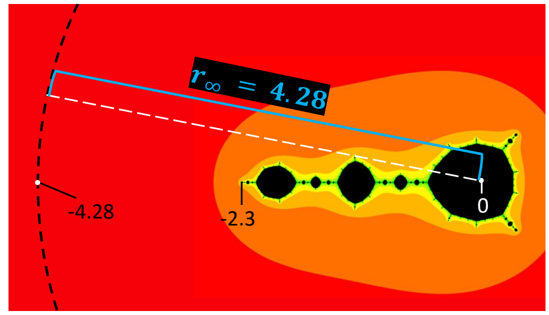

FIGURE 4. Escape radius for $f_{7.03,3,2}(z)$ calculated by Proposition 2.

We now use what we learned about the escape radius to study the behavior of the free critical point, $c_f = \frac{-md}{m+d}$. With Proposition 2, we fixed m, d, and most importantly, a, and determined which $z \in \mathbb{C}$ were in the basin of infinity. Now, however, we still fix m and d, but we focus on just the free critical point, $z = c_f$, and try to determine which a values in the parameter space put c_f in the basin of infinity.

Proposition 3. *If $d \geq m^m$ and $|a| > 2d\left(\frac{m+d}{md}\right)^m\left(\frac{m+d}{d}\right)^d$, then $|f^n_{a,m,d}(c_f)| \to \infty$ as $n \to \infty$.*

Proof. Choose d, m, and a such that $d \geq m^m$ and

$$|a| > 2d\left(\frac{m+d}{md}\right)^m\left(\frac{m+d}{d}\right)^d$$
$$= 2\left(\frac{d}{m^m}\right)\left(\frac{m+d}{d}\right)^{m+d}.$$

Since $d \geq m^m$, we can write

$$\frac{d}{m^m} \geq 1.$$

We also have $\left(\frac{m+d}{d}\right)^{m+d} > 1$, so it follows that $|a| > 1$.

Note from the proof of Proposition 2 that $2d$ is an escape radius.

Since $2d$ is an escape radius, if $|f_{a,m,d}(c_f)| > 2d$, then $|f^n_{a,m,d}(c_f)| \to \infty$ as $n \to \infty$. So we calculate $|f_{a,m,d}(c_f)|$:

$$\left| f_{a,m,d}\left(\frac{-md}{m+d}\right)\right| \tag{3}$$

$$= \left| a\left(\frac{-md}{m+d}\right)^m \left(1 + \frac{-md}{m+d}\left(\frac{1}{d}\right)\right)^d \right| \tag{4}$$

$$\geq |a| \left(\frac{md}{m+d}\right)^m \left(\frac{d}{m+d}\right)^d. \tag{5}$$

Since we have

$$|a| > 2d\left(\frac{m+d}{md}\right)^m \left(\frac{m+d}{d}\right)^d$$

we can substitute in Equation (5) to get

$$\left| f_{a,m,d}\left(\frac{-md}{m+d}\right)\right| > 2d.$$

Therefore, $|f_{a,m,d}(c_f)|$ is larger than an escape radius and by Proposition 2, $|f^n_{a,m,d}(c_f)| \to \infty$ as $n \to \infty$ when $d \geq m^m$. □

In Figure 5, we include an escape radius in the parameter space when $m = 3$ and $d = 27$. By Proposition 3, this bound on $|a|$ is given by $|a| > 47.2$. For every $|a| > 47.2$, the critical point c_f has an unbounded orbit. The black region, where the critical point has a bounded orbit, is completely contained in the disk of radius $|a| = 47.2$.

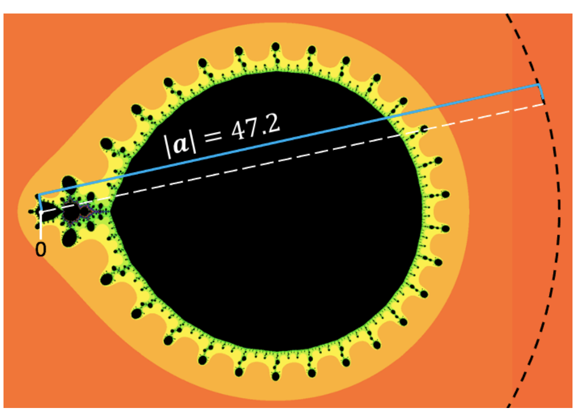

FIGURE 5. Escape radius in the parameter space of $f_{a,3,27}(c_f)$ calculated by Proposition 3.

Proposition 3 tells us $a, m,$ and d that send the free critical point to the basin of infinity as $n \to \infty$. We employ this proposition to propose the following corollary on connnectedness.

Corollary 5. *For $d \geq m^m$ and $|a| > 2d\left(\frac{m+d}{md}\right)^m \left(\frac{m+d}{d}\right)^d$, $K(f)$ is disconnected.*

Proof. By Proposition 3, when $d \geq m^m$ and $|a| > 2d\left(\frac{m+d}{md}\right)^m \left(\frac{m+d}{d}\right)^d$, $|f^n_{a,m,d}(c_f)| \to \infty$ as $n \to \infty$. Hence by Proposition 1, $K(f)$ is disconnected. □

Zero-Attracting Radius. We are also interested in determining when z maps to the basin of zero. The basin of zero contains all z such that $|f^n_{a,m,d}(z)| \to 0$ as $n \to \infty$. The basin of zero is used to define the zero-attracting radius, which we define below.

Definition 6. We define r_0 as a *zero-attracting radius* if when $|z| < r_0$, then $|f^n_{a,m,d}(z)| \to 0$ as $n \to \infty$.

Lemma 7. *Fix $r > 0$ and $k < 1$. Suppose $|f(z)| < k|z|$ for all $|z| < r$. Then $|f^n(z)| \to 0$ as $n \to \infty$.*

Proof. Let $r > 0$ and $k > 1$. Suppose $|f(z)| < k|z|$ for all $|z| < r$. Note that since $|z| < r$ and $k < 1$, we can write $|f(z)| < k|z| < kr < r$, or

$$|f(z)| < r. \tag{6}$$

Similar to the proof of Lemma 4, we now begin with induction to show that $|f^n(z)| < k^n|z|$ for all $n \geq 1$ and $|z| < r$. Our base case at $n = 1$ holds by assumption. Now assume $|f^n(z)| < k^n|z|$ for some $n \geq 1$ and all $|z| < r$. Consider the following

$$\begin{aligned} |f^{n+1}(z)| &= |f^n(f(z))| \\ &< k^n|f(z)| \text{by the induction} \\ &\quad \text{hypothesis and Equation 6} \\ &< k^n k|z| \\ &\quad \text{by the base case} \\ &= k^{n+1}|z|. \end{aligned}$$

Hence by induction $|f^n(z)| < k^n|z|$ for all $n \geq 1$ and $|z| < r$. Note that since $k < 1$, as $n \to \infty$, $k^n \to 0$. Thus by the Squeeze Theorem $|f^n(z)| \to 0$ as $n \to \infty$. □

Proposition 4. *Let* $r_0 = \min\{1, \frac{1}{2}\left(\frac{1}{|a|}\left(1+\frac{1}{d}\right)^{-d}\right)^{1/(m-1)}\}$. *Then for* $|z| < r_0$, $|f^n_{a,m,d}(z)| \to 0$ *as* $n \to \infty$.

Proof. Set $r_0 = \min\{1, \frac{1}{2}\left(\frac{1}{|a|}\left(1+\frac{1}{d}\right)^{-d}\right)^{1/(m-1)}\}$. We have

$$|f_{a,m,d}(z)| = |a||z|^m \left|1+\frac{z}{d}\right|^d$$
$$\leq |z|^{m-1}\left(1+\frac{|z|}{d}\right)^d |a||z|.$$

Define $g(x) = x^{m-1}\left(1+\frac{x}{d}\right)^d$ where $x = |z| \in \mathbb{R}^+$. We need $g(x) > 0$ and increasing for $x > 0$. Clearly $g(0) = 0$ and

$$g'(x) = x^{m-1}\left(1+\frac{x}{d}\right)^{d-1}$$
$$+(m-1)x^{m-2}\left(1+\frac{x}{d}\right)^d$$
$$= x^{m-2}\left(1+\frac{x}{d}\right)^{d-1}\left[x+(m-1)\left(1+\frac{x}{d}\right)\right]$$
$$> 0.$$

So $g(x)$ is in fact nonnegative and increasing for $|z| > 0$.

(i) Now suppose $|a| < \left(1+\frac{1}{d}\right)^{-d} < 1$. Since r_0 is defined as a minimum of two terms, we know $r_0 \leq 1$. If $|z| < r_0 \leq 1$,

$$|f_{a,m,d}(z)| \leq g(|z|)|a||z|$$
$$< \left(1+\frac{1}{d}\right)^d |a||z| < |z|,$$

so $g(|z|)|a| < 1$ and $|f^n_{a,m,d}(z)| \to 0$ as $n \to \infty$ by Lemma 7.

(ii) For $|a| \geq \left(1+\frac{1}{d}\right)^{-d}$, take

$$r_0 \leq \frac{1}{2}\left(\frac{1}{|a|}\left(1+\frac{1}{d}\right)^{-d}\right)^{1/(m-1)}$$
$$< \frac{1}{2}\left(\left(1+\frac{1}{d}\right)^d\left(1+\frac{1}{d}\right)^{-d}\right)^{1/(m-1)}$$
$$= \frac{1}{2} < 1.$$

If $|z| < r_0$, then $|f_{a,m,d}(z)| \leq g(|z|)|a||z|$ and

$$g(|z|)|a| \leq r_0^{m-1}\left(1+\frac{r_0}{d}\right)^d |a|$$
$$< \frac{1}{2^{m-1}|a|}\left(1+\frac{1}{d}\right)^{-d}\left(1+\frac{r_0}{d}\right)^d |a|$$

Let $j(x) = \left(1+\frac{x}{d}\right)^d$. Since $j(x)$ is positive and increasing for $x > 0$, if $r_0 < 1$, then $j(r_0) < j(1)$ means $\left(1+\frac{r_0}{d}\right)^d < \left(1+\frac{1}{d}\right)^d$, which we can write as $\left(1+\frac{1}{d}\right)^{-d}\left(1+\frac{r_0}{d}\right)^d < 1$. Hence, $g(|z|)|a| < 1$ for $|z| < r_0$ and $|f^n_{a,m,d}(z)| \to 0$ as $n \to \infty$ by Lemma 7. □

Figure 6 presents the zero-attracting radius, r_0, for $a = 0.39$, $m = 2$ and $d = 3$. The radius r_0 is calculated by way of Proposition 4, as

$$r_0 = \min\{1, \frac{1}{2}\left(\frac{1}{|a|}\left(1+\frac{1}{d}\right)^{-d}\right)^{1/(m-1)}\}$$
$$= \min\{1, 0.54\} = 0.54.$$

By Definition 5.3, every z in the disk of radius 0.54 maps to zero as $n \to \infty$.

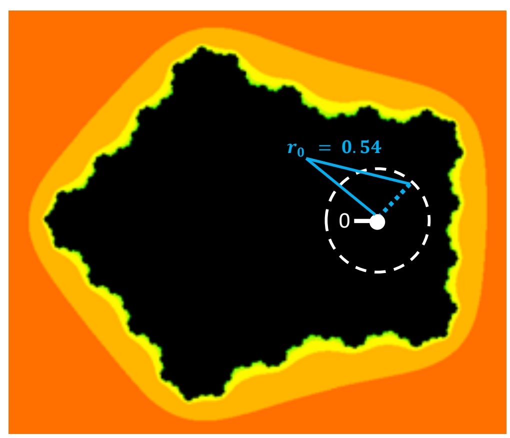

FIGURE 6. The zero-attracting radius for $f_{0.39,2,3}$ calculated by Proposition 4.

Now that we have an attracting radius, we consider parameter values that put our free critical point in the basin of zero.

Proposition 5. *If* $|a| < \min\{\left(\frac{md}{m+d}\right)^{-m}, \left(1+\frac{1}{d}\right)^{-d}\}$, *then* $|f^n_{a,m,d}(c_f)| \to 0$ *as* $n \to \infty$.

Proof. Recall from Proposition 6 that 1 is a zero-attracting radius. Therefore, if we want $|f^n_{a,m,d}(c_f)| \to 0$ as $n \to \infty$, we can show that $|f_{a,m,d}(c_f)| < 1$. Since $|a| < \left(\frac{md}{m+d}\right)^{-m}$, we have

$$\begin{aligned}
&|f_{a,m,d}(c_f)| \\
&= \left| a\left(\frac{-md}{m+d}\right)^m \left(1 + \frac{-md}{m+d}\left(\frac{1}{d}\right)\right)^d \right| \\
&= |a| \left(\frac{md}{m+d}\right)^m \left(\frac{d}{m+d}\right)^d \\
&< \left(\frac{d}{m+d}\right)^d < 1.
\end{aligned}$$

Hence by Proposition 6, $|f^n_{a,m,d}(c_f)| \to 0$ as $n \to \infty$. □

In Figure 7, we include a zero-attracting radius in the parameter space for $m = 2$, $d = 2$. We use Proposition 5 to calculate this bound on $|a|$, with

$$\begin{aligned}
|a| &< \min\{\left(\frac{md}{m+d}\right)^{-m}, \left(1 + \frac{1}{d}\right)^{-d}\} \\
&= \min\{1, 0.44\} = 0.44.
\end{aligned}$$

Every $|a| < 0.44$ has an orbit that converges to zero as $n \to \infty$.

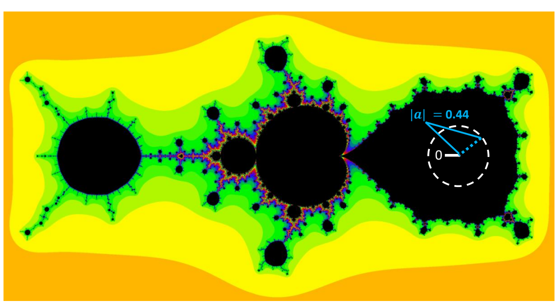

FIGURE 7. The zero-attracting radius in the parameter space of $f_{a,2,2}(c_f)$ calculated by Proposition 5.

Proposition 5 tells us $a, m,$ and d that send the free critical point to the basin of zero as $n \to \infty$. We employ this proposition to propose the following corollary on connnectedness.

Corollary 8. *For* $|a| < \min\{\left(\frac{md}{m+d}\right)^{-m}, \left(1 + \frac{1}{d}\right)^{-d}\}$, *$K(f)$ is connected.*

Proof. By Proposition 5, when $|a| < \min\{\left(\frac{md}{m+d}\right)^{-m}, \left(1 + \frac{1}{d}\right)^{-d}\}$, we have $|f^n_{a,m,d}(c_f)| \not\to \infty$ as $n \to \infty$, because $|f^n_{a,m,d}(c_f)| \to 0$ as $n \to \infty$. Hence by Proposition 1, $K(f)$ is connected. □

DISCUSSION

We have now analyzed the behavior of our free critical point through a variety of lenses. First we focused on the connectedness of the filled Julia set for different values of the parameter a. We found with Proposition 1 that when a is in a colored region of the parameter space, $K(f)$ is disconnected, and when a is in a black region of the parameter space, $K(f)$ is connected. Next we determined an escape radius, where z larger than this radius are guaranteed to be in the basin of infinity. We also developed an escape radius in the parameter space, which gives distinct parameter values that put our critical point in $A_\infty(f)$. Finally, we determined a radius within which all z values go to zero, as well as a radius in the parameter space, within which the free critical point goes to zero. With future work, we hope to further analyze the behavior of our family of functions and their convergence to a family of transcendental maps as $d \to \infty$.

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