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ABSTRACT In this project we studied the mathematical concept of the Frobenius number and some curious patterns that come with it. One common example of the Frobenius number is the Coin Problem: If handed two denominations of coins, say 4¢ and 5¢, and asked to create all possible values, we will eventually find ourselves in a position where we can make any value. With 4¢ and 5¢ coins, we can create any value above 11¢, but not 11¢ itself. So, that makes 11 the Frobenius number of 4 and 5. What we explore in this paper is a pattern we call Frobenius symmetry: when all non-negative integers below the Frobenius number can be paired up such that one number is attainable, and the other is now. We looked at sets of two and three numbers and arrived at results about both.

INTRODUCTION

The core of our research is the concept of the Frobenius number and the easiest way to explain it is through the Coin Problem.

In the Coin Problem, you are handed two denominations of coins. For example, you can have 3¢ and 5¢ coins. Then there are certain monetary values that you can and cannot create using just those two denominations of coins. It is clear that you can create 3¢, 5¢, 6¢, 8¢, 9¢, or 10¢ quite easily. These values we call attainable. However, no matter how you combine 3¢ and 5¢ coins, you can never create the values of 1¢, 2¢, 4¢, or 7¢, and thus they are unattainable.

An interesting variant of this problem to consider is when you can receive change from the denominations you are using. So, with 3¢ and 5¢ coins, giving two 3¢ to buy something for 5¢ results in you getting 1¢ back, in which case 1¢ is now an attainable value. As a result of Number Theory, this will work every time the two coins you start with have no common factor.

Now let us return to the original problem. In the example of 3¢ and 5¢ coins, 7¢ is the largest monetary value that is unattainable. This can be seen by noting that by repeatedly adding 3¢ coins to the values of 8¢, 9¢, or 10¢, values that we know are attainable, we can attain every following number. 7¢, as the largest unattainable value when you start with 3¢ and 5¢ coins, is thus the Frobenius number of 3¢ and 5¢. Mathematically, we would say that 7 is the

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Frobenius number of {3, 5}. This is displayed below in Table 1.

0	0*3 + 0*5	6	2*3 + 0*5
1	Unattainable	7	Unattainable
2	Unattainable	8	1*3 + 1*5
3	1*3 + 0*5	9	3*3 + 0*5
4	Unattainable	10	0*3 + 2*5
5	0*3 + 1*5	…	(8,9,10) + x*3

Table 1. Each number on the left is followed by the combination used to attain it.

An interesting pattern to observe in Table 1 is that the numbers, as they grow, form a neat symmetry: Attainable, Unattainable, U, A, U, A, A, U, A from 0 to 7. As displayed in Table 2, below, when we paired each number with the value attained by subtracting it from 7, the paired values had opposite attainability. This pattern we found to repeat in other cases and is the primary pattern that our research focused on.

k	F - k
0 – A	7 – U
1 – U	6 – A
2 – U	5 – A
3 – A	4 – U

Table 2. 'A' stands for attainable and 'U' stands for unattainable. Pairs are seen across the vertical line.

Note that the Frobenius number is calculable from a set of two numbers. If you begin with a set {*a*, *b*} where *a* and *b* have no common factors other than 1, the Frobenius number *F* equals *ab-a-b*. The reason that *a* and *b* cannot share any common factors is because otherwise the attainable values will be restricted to multiples of that common factor, meaning there is no highest unattainable value. In other words, *a* and *b* have to be relatively prime.

We also looked at the Frobenius numbers of sets containing three numbers. There is no formula to plug numbers in to like there is for sets with two numbers. However, looking at the set {6, 9, 20} we can calculate the Frobenius number to be 43. The same restriction on having no common factors for the entire set continues into sets of three numbers as well.

As an example of this, looking at 3¢ and 5¢ again, if we add in a 10¢ coin, the Frobenius number remains 7. But, if we add in a 4¢ coin, the Frobenius number is 3 and the table turns into what we see in Table 3.

k	F - k
0 – A	3 – U
1 – U	2 – U

Table 3. 'A' stands for attainable and 'U' stands for unattainable. Pairs are seen across the vertical line.

The dichotomous pairing is gone, despite there being a Frobenius number.

Another interesting case to observe is when you start with {6, 9, 20}. In this case, given just 6 and 9, or 6 and 20, there would be no Frobenius Number, as those pairs of values could only produce multiples of 3 or 2 respectively and will therefore always have gaps between any numbers they produce. A third value allows for a Frobenius number in this case, and the expected pairing appears as well.

All of these variant cases interested us enough to create the definition of *Frobenius symmetry*:

> Let *F* be the Frobenius number of a set of numbers {a, b, c, …}.
>
> We say that {a, b, c, …} has *Frobenius symmetry* if for all *k* with $0 \leq k \leq F$, *k* is attainable if and only if (*F*-*k*) is unattainable.

In this project, we proved that Frobenius Symmetry holds for every set of two numbers that are relatively prime and collected data to establish conjectures for sets of three numbers.

METHODS

Our main method of gathering data and finding patterns was Maple, a mathematics-based programming language. We built codes that would test sets of two or three numbers for

Frobenius symmetry and used the results to build the database on which we based our conjectures for sets of three.

Using the data from Maple, we built formal mathematical proofs about relevant sets of two numbers and formed conjectures about sets of three numbers.

RESULTS

Our primary result is a proof of Frobenius Symmetry for all sets of two numbers where the numbers are relatively prime. First, a formal definition of attainable numbers:

Definition. *Attainable Numbers* - Given two relatively prime natural numbers a and b, an attainable number is any non-negative integer that can be written as $ap+bq$, where p and q are non-negative integers.
From just this definition we have the main theorem of Frobenius symmetry:

Theorem 5. Given a and b relatively prime, if $0 < k < F$, then k is attainable if and only if its partner (F-k) is unattainable.

What we want to see at the end of this proof was that each number between 0 and F has a partner number that had the opposite attainability. To begin, we need to reorganize the number line to make it easier to work with. Organize the positive integers into a table like Table 4 where the number of rows was equal to the smaller of the two numbers letting $a < b$.

0	a	2a	3a	…
1	a+1	2a+1	3a+1	…
…	…	…	…	…
a-1	2a-1	3a-1	4a-1	…

Table 4. Template for organizing all positive integers.

Note that once a number becomes attainable in any row, every number to the right of that attainable value is also attainable by repeated adding of a. Our next focus was then the first attainable number of each row.

Definition. *Mod a* - We say that two numbers b and c are congruent, or equal, mod a if they leave the same remainder upon division by a. This is denoted $[c]_a = [b]_a$.

Remark. If we number the rows from 0 to a-1, and let s be the row number, every number in a row s is equal to s mod a.

Theorem 1. If a and b are relatively prime natural numbers and all non-negative numbers are arranged into an array with a rows, then the first attainable number in each row will be of the form qb where $q = 0, 1, \dots, (a\text{-}1)$.

Proof. Let a, b be relatively prime natural numbers. Let all non-negative integers be arranged into an array with a rows as shown in Table 4. Let s be the row number going from 0 to a-1.

The first attainable in every row we know to be a multiple of b because:

- Adding or subtracting a to an element of a row will result in an element in the same row since the array has a rows.
- By definition, all attainable numbers take the form of $ap + bq$ where p, q are non-negative integers. If an attainable number has a $p > 0$, then subtracting ap will result in some number a*0 + ba, which will be a multiple of b, attainable and in the same row.

Since there are a rows, the first attainable number in each row will have some $q = 0, 1, \dots, a\text{-}1$. q has to be within this range because if $q < 0$ then it would result in a negative first attainable number, and if $q \geq a$ then you would achieve some first attainable plus a multiple of a.

Now that we know how each row was split between attainable numbers and unattainable ones, we need a way to pair numbers.

Definition. *Paired Rows* – Let s_1 and s_2 be two rows within the array. Two rows are said to be paired if: $[s_1]_a + [s_2]_a = [F]_a$.

Theorem 2. The first attainable number in any row plus the last unattainable in its paired row add up to the Frobenius number.

Proof. By the definition of Row Number, we know that any first attainable number qb equals s mod a. Solving for q, we get q equals sb^{-1} mod a.

Let r be the smallest nonnegative element of $[sb^{-1}]_a = [q]_a$. So, the first attainable number in row s can also be written as rb. The last unattainable number in any row s is $(rb - a)$ because the array has a rows.

Let s_1 and s_2 be paired rows. Let r_1 be the remainder mod a when s_1b^{-1} is divided by a and let r_2 be the remainder mod a of s_2b^{-1} is divided by a. The sum of the first attainable number in one row and the last unattainable number in its paired row is therefore:

$$r_1b + (r_2b - a),$$

which can be simplified to:

$$b(r_1 + r_2) - a.$$

Before we can finish the proof of this theorem, we need some way to understand the value $r_1 + r_2$. Since we want that last expression to be F, we use the following lemma to show how $r_1 + r_2 = a - 1$, the value that would get us what we need.

Lemma 3. $r_1 + r_2 = a - 1$.

Proof. We know that $[r]_a = [sb^{-1}]_a$. We can add the congruence classes of two paired rows together:

$$[s_1b^{-1}]_a + [s_2b^{-1}]_a$$

$$= [s_1b^{-1} + s_2b^{-1}]_a$$

$$= [b^{-1} (s_1 + s_2)]_a.$$

Of the set of all first attainable numbers, $[0*b, 1*b, \ldots, (a - 1)*b]$, $(a - 1)*b$ is the largest element of the set and is an element of the congruence class $[-b]_a$ because:

$$[(a - 1)*b]_a = [ab - b]_a = [-b]_a.$$

So, since s_1 and s_2 are paired rows, their sum mod a equals the Frobenius number mod a, which is equal to $-b$ mod a …

$$= [b^{-1} (s_1 + s_2)]_a$$

$$= [b^{-1}]_a[(s_1 + s_2)]_a$$

$$= [b^{-1}]_a[-b]_a$$

$$= [b^{-1}(-b)]_a$$

$$= [-(b^{-1}b)]_a$$

$$= [-1]_a.$$

Therefore,

$$[r_1]_a + [r_2]_a = [-1]_a = [a-1]_a.$$

So, the lemma is true mod a.

However, we are looking for the specific value of $(a- 1)$. The possible options for $r_1 + r_2$ are:

$$[\ldots, -1, a–1, 2a-1, 3a-1, \ldots].$$

Since r_1 and r_2 are remainders mod a, then their sum cannot be negative, which rules out -1 and below.

For the same reason, the largest their sum will ever be is $(2a-2)$, so $(2a-1)$ and above can be ruled out too. Therefore, the only possible sum of $r_1 + r_2$ is $a - 1$.

Now we can return to the proof of Theorem 2:

The sum of the first attainable number in one row plus the last unattainable in its paired row is $b(r_1 + r_2)-a$ by our work before Lemma 3. By the lemma, though, the sum is $b(a-1)-a$, which simplifies down to $ab-a-b$, the Frobenius number, proving Theorem 2.

We have gone from understanding how elements are paired to understanding how rows are paired. Now we need to show that the distribution of attainable versus unattainable numbers is exactly half and half.

Theorem 4. The total number of unattainable elements is equal to the number of attainable elements less than and equal to the Frobenius number.

Proof. Let s_1 and s_2 be paired rows. By our definition of paired rows, each element in s_1 has a unique partner in the paired row s_2. Let P be a function $P(t) = F - t$. Table 5 is an example of the set {4, 7}.

0	**4**	**8**	**12**	**16**	**20**	**…**
1	5	9	13	*17*	**21**	**…**
2	6	10	**14**	**18**	**22**	**…**
3	**7**	**11**	**15**	**19**	**23**	**…**

Table 5. Bolded values are attainable while underlined values are unattainable.

Looking at just the last two rows…

S_1	S_2
2	**15**
6	**11**
10	**7**
14	3

Table 5. The left column represents values of t, while the right is $F - t$.

So, let x be the column position of the first attainable in s_1, let y be the column position of the last unattainable in s_2, and let z be the total number of elements less than or equal to F in a row, with the leftmost column being Column 0. As an example, looking at the third row, the first attainable number 14 is in Column 3 of the array while the last unattainable number is 10 in Column 2.

The number of unattainable elements to the left of the first attainable in s_1 will be x, which in our example is 3. The number of unattainables in row s_2 will be $(y + 1)$. This is because the first attainable in this row will be in the next column over, $(y + 1)$. Therefore, the total number of unattainables between two paired rows is $x + (y + 1)$.

Now we want to show that this is exactly half of the total number of elements.

There are z elements in each row, so if we start at 0 the columns will go from 0 to z - 1. As seen in Table 4, the farthest left element of one row will be paired with the farthest right element of the other row, after which the pairs move sequentially towards the other end. This means that the column positions of any pair will always add up to $(z - 1)$.

At some point, we will reach the pair of the first attainable in one row and the last attainable in the other, which have column positions x and y respectively.

Thus, $x + y = z - 1$, i.e., $x + y + 1 = z$.

Logically, $x + y + 1 = (½)(2z)$ and so we have that exactly half of the elements between two paired rows will be unattainable.

Therefore, since every row is paired, and every pair of rows has exactly half of its elements attainable, then of all numbers less than or equal to F, exactly half will be attainable.

This is the main result that we needed. We now knew that exactly half of the numbers between 0 and F are unattainable, the others attainable, and we have a system of pairing them up. We are ready to prove Frobenius symmetry.

Theorem 5. Given a and b relatively prime, if $0 < k < F$, then k is attainable if and only if its partner $(F\text{-}k)$ is unattainable.

Proof. We know that exactly half of the numbers from 0 to F are attainable, and the other half are not. We also know that every number has a partner in a paired row where the two add up to the Frobenius number.

Want: We want to show that every attainable number has an unattainable partner.

The three situations that we could have for any specific pair (k, F-k) would be:

- Attainable + Attainable
- Attainable + Unattainable
- Unattainable + Unattainable

The first cannot happen because two attainable numbers adding up to the Frobenius number would mean that the Frobenius number is attainable, causing a contradiction.

The second situation is what we want.

The third situation is similarly impossible. If two unattainable numbers are paired together, then due to the equal number of unattainable and attainable numbers, there would have to be two attainable numbers that would be paired with each other whose sum would be the Frobenius number, which we have already shown is impossible.

Therefore, any element k is attainable if and only if its partner (F-k) is not, proving the main theorem.

What this means is that any set of two numbers, as long as they are relatively prime, have Frobenius symmetry.

Results regarding sets of 3 numbers

This concluded our proof of Frobenius symmetry for two relatively prime numbers, but we also formed conjectures about sets of three numbers.

Conjecture 1. If we have some set $\{a, b, c\}$ where c is an attainable of the $\{a, b\}$, Frobenius symmetry holds.

Evidence. {3, 5, 6} – Anything that can be attained by adding 6 can be attained by adding 3 twice, so nothing new is gained by adding in 6.

Conjecture 2. If we have some set $\{a, b, c\}$ where c is the Frobenius number of $\{a, b\}$, Frobenius symmetry fails.

Evidence. Our database contains data points from {3, 4, 5} to {11, 13, 119} of sets where the third number is the Frobenius number of the first two. Frobenius symmetry always fails.

Conjecture 3. If we have some set $\{a, b, c\}$ where a is relatively prime with b, b is relatively prime with c, and c is relatively prime with a, Frobenius Symmetry fails.

Evidence. One of the Maple codes we built checked every set of three numbers from (1, 2, 3) to (98, 99, 100) and found that Frobenius symmetry failed in every case where the three numbers were all relatively prime to each other.

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