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Strong Hanani-Tutte on the Projective Plane

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Abstract

If a graph can be drawn in the projective plane so that every two non-adjacent edges cross an even number of times, then the graph can be embedded in the projective plane.

1 Introduction

In the plane there is a beautiful characterization of planar graphs known as the Hanani-Tutte theorem: a graph is planar if and only if it can be drawn in the plane so that every two non-adjacent edges cross an even number of times. Equivalently, any drawing of a non-planar graph in the plane must contain two non-adjacent edges that cross oddly.

There are several proofs of the Hanani-Tutte theorem, including the original 1934 proof by Hanani and the 1970 proof by Tutte, see [7] for more references. Our goal in the current paper is to show that the result remains true in the projective plane.[1]

Theorem 1.1. *Let G be a graph. Suppose that G can be drawn in the projective plane so that every two non-adjacent edges cross evenly. Then G can be embedded in the projective plane.*

This is not the first result that indicates that the Hanani-Tutte theorem is not a special property of the plane. Using homology theory, Cairns and Nikolayevsky [2] showed that if a graph can be drawn on an orientable surface

[1] A sphere with a crosscap. We assume that the reader is familiar with the basic terminology of drawings and embeddings in surfaces. For background see [6, 3].

so that every pair of edges (not just non-adjacent ones) crosses an even number of times, then the graph can be embedded in that surface. Pelsmajer, Schaefer, and Štefankovič [8] gave a new, elementary proof of this weak Hanani-Tutte theorem that also establishes the result for non-orientable surfaces. Theorem 1.1 is the first time the strong version of the Hanani-Tutte theorem has been established for any higher-order surface.

There is an alternative view of the Hanani-Tutte theorem in terms of crossing numbers. The *crossing number* of a graph G, denoted by $\mathrm{cr}_S(G)$, is the minimum number of pairs of edges that cross in any drawing of G in surface S. Hence a graph G is embeddable in S if and only if $\mathrm{cr}_S(G) = 0$. The *odd crossing number* of G, denoted by $\mathrm{ocr}_S(G)$, is the minimum number of pairs of edges that cross oddly in any drawing of G in surface S. The *independent odd crossing number* of G, $\mathrm{iocr}_S(G)$, is the minimum number of pairs of non-adjacent edges that cross oddly in any drawing of G in surface S.

The strong Hanani-Tutte theorem can now simply be stated as "$\mathrm{iocr}(G) = 0$ implies $\mathrm{cr}(G) = 0$" and Theorem 1.1 becomes "$\mathrm{iocr}_{N_1}(G) = 0$ implies $\mathrm{cr}_{\mathrm{N}_1}(G) = 0$" using N_1 as a symbol for the projective plane. The weak Hanani-Tutte theorem in this notation reads "$\mathrm{ocr}_S(G) = 0$ implies $\mathrm{cr}_S(G) = 0$" and is true for all surfaces S as we mentioned above. The crossing number point of view emphasizes the algebraic nature of the Hanani-Tutte theorem as argued by van der Holst in [9].

Our proof of the strong Hanani-Tutte theorem for the projective plane uses techniques we developed for the Hanani-Tutte theorem and related results in the plane and higher-order surfaces [7, 8] and combines them with ideas from Mohar and Robertson on embeddings in the projective plane [5]; see Section 2. The proof will not naturally extend to any surface other than the projective plane, since it makes use of the list of minimal forbidden minors for the projective plane.

2 From Embeddings to Drawings

In this section we develop the necessary tools to deal with drawings in the projective plane. Some of these tools are extensions of well-known results for embeddings. All of them will play an important rôle in the proof of the strong Hanani-Tutte theorem for the projective plane.

2.1 Basic Observations

Recall that a closed curve is *contractible* if it can be contracted to a point. In the projective plane a closed curve is contractible if and only if it passes through the crosscap an even number of times.

Lemma 2.1. *The family of non-contractible cycles in a graph drawn in the projective plane satisfies the* 3*-path condition: given three internally disjoint paths with the same endpoints, if two of the cycles formed by the paths are contractible then so is the third.*

Proof. Let P_1, P_2, P_3 be the three paths. Call a path even (odd) if it passes through the crosscap an even (odd) number of times. Then a cycle is contractible if and only if it is formed by two paths of the same parity. If two of the cycles are contractible then all three paths have the same parity and the third cycle is also contractible. □

Lemma 2.1 is based on [6, Proposition 4.3.1].

For convenience, we say that a particular drawing of a graph is *iocr*-0 if no pair of non-adjacent edges crosses an odd number of times.

Lemma 2.2. *If a graph G drawn on the projective plane contains two vertex-disjoint non-contractible cycles, then the drawing is iocr-0.*

Proof. In the projective plane any two non-contractible curves cross an odd number of times.[2] Therefore there must be an edge in each of the two cycles such that the two edges cross oddly. These must be non-adjacent, as they belong to vertex-disjoint cycles, so the given drawing of G is iocr-0. □

A ΔY*-exchange* in G is a process that replaces a triangle in a drawing of G with a claw (a $K_{1,3}$). The three vertices of the triangle become the leaves of the claw.

Lemma 2.3. *Let G be a graph with $\mathrm{iocr}_{N_1}(G) > 0$, and suppose G' can be obtained from G by a ΔY-exchange. Then $\mathrm{iocr}_{N_1}(G') > 0$.*

Proof. Consider an iocr-0 drawing of G'. Let e_1, e_2 and e_3 be the three edges of the claw. Draw a new edge f_1 by closely following e_1 and e_2; similarly add f_2 following e_2, e_3 and f_3 following e_3, e_1. If f_1 crosses an edge e of $G' - \{e_1, e_2, e_3\}$ oddly, then e must cross either e_1 or e_2 oddly; hence e is incident to f_1. Similarly f_2 and f_3 only cross adjacent edges oddly. Removing e_1, e_2, e_3 now yields an iocr-0 drawing of G, which implies that $\mathrm{iocr}_{N_1}(G') = 0$. □

2.2 Redrawing Tools

We will occasionally apply redrawing moves that lead to self-intersections of edges. These can be removed as shown in Figure 1.

The removal of self-intersections does not change the type of a curve in the projective plane:

Lemma 2.4. *If C is a closed curve drawn in the projective plane, and C' is a closed curve obtained from C by removing all self-intersections as shown in Figure 1, then C is contractible if and only if C' is; moreover, C' is a* simple *closed curve.*

[2]Any one-sided (or non-contractible) curve can serve as the crosscap, and we know that the other curve must use the crosscap an odd number of times since it is non-contractible.

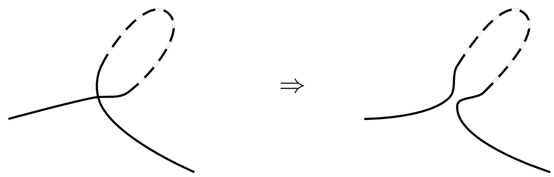

Figure 1: Removing a self-intersection; illustration from [7].

Proof. Let D be the boundary of the crosscap; then D is a simple closed curve. Any closed curve in the projective plane can be modified slightly so that it crosses D a finite number of times, and then it is contractible if and only if it crosses D an even number of times. The crossing parity[3] between the two curves is not changed by removing self-intersections. □

We will use the following lemma in Section 3.1 to clear a cycle in $K_{3,5}$ of crossings. The proof is based on ideas from [7, Theorem 3.1]. An edge in a drawing is *even* if it crosses every other edge an even number of times.

Lemma 2.5. *Let G be a graph with $\mathrm{iocr}_{N_1}(G) = 0$, and let C be a non-contractible cycle in an iocr-0 drawing of G. Then G can be redrawn so that the independent odd crossing number remains zero and no edge of C is involved in any crossing.*

Proof. By redrawing locally near each vertex of C, we can make all edges of C even, as follows: For any two consecutive edges e, e' incident to a common vertex v, we redraw e near v (if needed) so that e and e' cross an even number of times. Then for every other edge f incident to v, we redraw f near v so that f crosses each of e and e' evenly. Since the original drawing is iocr-0, all edges on C are now even and the new drawing is still iocr-0.

Now, contract all the edges of C but one, call it e_C. The edge e_C is now an even loop, possibly with self-intersections, which we can remove as shown in Figure 1. By Lemma 2.4, e_C is non-contractible, since C is.

For each edge f that crosses e_C, partition the crossings into pairs along e_C. Each such pair splits e_C into two curves; let c be the one that avoids the endpoint of e_C. Severing f at e_C turns each pair of crossings into four free ends, which we reconnect by two curves along each side of c; see Figure 2.2, left and middle. Repeating this for all edges crossing e_C removes all crossings with e_C, while preserving crossing parity for every pair of edges. However, an edge f might turn into a set of curve components. Consider a component of f and deform a small portion of it, without crossing e_C, so that it joins another component of f; this is possible since e_C is not surface-separating (because e_C is non-contractible); see Figure 2.2, middle and right. This deformation does not change the crossing parity between any two edges, and repeating this process turns f into a simple curve again.

[3] The *crossing parity* of two curves or edges is the parity of the number of times they cross.

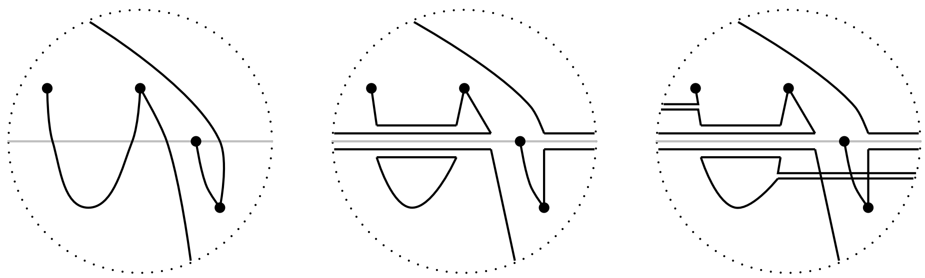

Figure 2: Cycle C contracted to a single edge e_C (gray) with two crossing edges (black) in the projective plane (crosscap as dotted line). Before redrawing *(left)*; after severing edges crossing e_C and reconnecting ends along e_C *(middle)*, note that both curves consist of two components now; after reconnecting the two components of the left curve *(right)*.

Finally, we can uncontract to recover C from e_C, because we did not change the rotation at the vertex of e_c. Now C is a crossing-free non-separating cycle in an iocr-0 drawing of G. □

The following lemma shows, roughly speaking, that for an iocr-0 drawing it is not a single vertex that makes the difference between planarity and non-planarity.

Lemma 2.6. *Let $x \in V(G)$ and $H = G - x$. Suppose there is an iocr-0 drawing of G in the projective plane such that all cycles in H are contractible. Then G is planar.*

Proof. Consider the specified drawing of G in the projective plane.

Claim: We can redraw G so that each edge of H passes through the crosscap an even number of times, and the drawing is still iocr-0.

Let F be a spanning forest of H. Process the edges of F in a breadth-first order as follows: suppose uv is an edge of F oriented towards the root of its component (that is, u is closer to the root of uv's component than v). Contract uv by moving v along uv towards u, pushing all crossings along with it until uv passes through the crosscap an even number of times. Call uv *processed*. Note that this move does not change the parity of how often any processed edge of F other than uv passes through the crosscap, since the only edges whose parity is changed by the contraction are edges incident to v, and none of those can have been processed already. At the end, every edge of F passes through the crosscap an even number of times. Every edge in $E(H) - E(F)$ also uses the crosscap evenly, since it completes a contractible cycle with some edges in F.

We now remove the crosscap and replace it with a disk. We reconnect severed edges by simple curves within the newly added disk.

Any two such curves within the disk have to cross oddly, since their crossings with the disk boundary alternate. Since each edge of H passes through the disk an even number of times, its crossing parity with every other edge does not change. Hence, any two edges whose crossing parity changes must be incident to x, which means that the independent odd crossing number is not affected by replacing the crosscap with a disk. We have thus obtained an iocr-0 drawing of G in the sphere (and, thereby, the plane), which implies that the graph is planar by the Hanani-Tutte theorem for the plane. □

2.3 K-graphs and iocr

Let H be a subgraph of a graph G. An *H-component* or *H-bridge* is either an edge (and its endpoints) that does not belong to H but both of whose endpoints do, or a connected component of $G - V(H)$ together with all edges (and their endpoints) connecting this component to H.

H is a *K_4-graph* of G if it is a subdivision of K_4, and there is an H-component that is attached to all the vertices of degree 3 in H. H is a *$K_{2,3}$-graph* of G if it consists of three internally disjoint paths connecting two vertices, x and y, and there is an H-component that contains at least one internal vertex of each of the three paths. A *K-graph* is either a K_4-graph or a $K_{2,3}$-graph.

The following result is a well-known fact for embeddings (see [5, p326]). We relax the assumption that G is embedded to allow crossings, but control the crossings by requiring $\text{iocr}_{N_1}(G) = 0$.

Lemma 2.7. *Let G be a graph containing a K-graph. Then every iocr-0 drawing of G in the projective plane contains two non-contractible cycles in the K-graph.*

Proof. Fix a drawing of G with independent odd crossing number 0 in the projective plane; let H be the K-graph within G. Note that if a K-graph contains one non-contractible cycle, then it must contain two (this is a consequence of the 3-path property, Lemma 2.1). Hence, for a contradiction, we may assume that all the cycles in H are contractible.

Suppose first that H is a subdivision of a $K_{2,3}$. Then H is the union of three paths P_1, P_2, P_3 that have the same endpoints x and y but share no other vertices, and there is an H-component B containing an internal vertex of each path. Consider three edges from these vertices of H to $V(B) - V(H)$, and let T be a minimal tree in B that contains those edges. Then T has three leaves, so it must contain a unique vertex z of degree 3. Let $G' = H \cup T$. By construction, G' is a subdivision of $K_{3,3}$. The drawing of G yields an iocr-0 drawing of G'. All the cycles in $G' - z$ are cycles in H, which are contractible by assumption. Applying Lemma 2.6 implies that G' is planar, which is a contradiction.

If H is a subdivision of a K_4, let S be the set of its 4 vertices of degree 3 and let B be an H-component that contains S. As above, there is a tree T in B such that the set of its leaves is S. Since T has four leaves it either has a unique vertex of degree 4 or two vertices of degree 3. Let $G' = H \cup T$ which is a minor of K_5. If T contains a vertex z of degree 4 we proceed as in the case of $K_{2,3}$: the cycles in $G' - z$ are cycles in H and therefore contractible, but then

Lemma 2.6 implies that G' is planar, which is a contradiction. If T contains two vertices u, v of degree 3, let Q be the path between u and v in T. Let uw be the first edge in the path (possibly $w = v$). Contract uw by moving u along uw to w and then identifying u and w. As we contract, we may create odd crossings and self-intersections. Self-intersections we deal with using Lemma 2.4. Any new odd crossing will be between an edge e_1 that was incident to u before the contraction, and an edge e_2 that had crossed uw oddly. But e_2 must have been incident to either u or w, since the drawing was iocr-0. After contraction both edges are incident to $u = w$, so the drawing is still iocr-0. In this fashion we can contract all the edges along the path Q until we have a single vertex of degree 4 and are back in the first case, which suffices. □

Apart from applying Lemma 2.7 directly, we will also use it in the two variants stated as corollaries below.

Corollary 2.8. *Let G be a graph containing a K-graph. Then every iocr-0 drawing of G in the projective plane contains a non-contractible induced cycle.*

Proof. By Lemma 2.7 the drawing of G contains a non-contractible cycle, so we can choose a shortest non-contractible cycle. If that cycle had a chord, that chord would split the cycle into two shorter cycles which are therefore contractible. But by the 3-path property this would imply that the cycle itself would have to be contractible, a contradiction. □

Corollary 2.9. *If a graph G contains two disjoint K-graphs, then* $\text{iocr}_{N_1}(G) > 0$.

Proof. Assume $\text{iocr}_{N_1}(G) = 0$. Applying Lemma 2.7 twice gives us two vertex-disjoint non-contractible cycles, which contradicts Lemma 2.2. □

3 Proof of the Main Theorem

If G cannot be embedded in the projective plane, then it must contain at least one of 35 minimal forbidden minors for the projective plane determined by Archdeacon, Glover, Huneke, and Wang [1, 4]. For a complete list of minimal forbidden minors and their names see [6, p198] or [5][4]. We will show that all these graphs have independent odd crossing number larger than zero, which will establish Theorem 1.1. It suffices because given an iocr-0 drawing of a graph, one can easily obtain an iocr-0 drawing of any minor of that graph.[5]

The first twelve graphs are formed from two Kuratowski graphs by a disjoint union, a one-vertex identification, or a two-vertex identification and possibly deleting an edge between these vertices. It is easy to see that each contains two disjoint K-graphs. By Corollary 2.9 the independent odd crossing number of each of these twelve graphs is nonzero.

[4] All reference to [5] in this section are to the proof of Theorem 3.1 in that paper.

[5] We already used this at the end of the proof of Lemma 2.7. This fact also underlies almost all proofs of the strong Hanani-Tutte theorem.

Of the remaining 23 minimal forbidden minors, C_7, E_{19}, D_{12}, E_{11}, E_{27}, D_9, G_1 [5, Fig. 3] and D_{17}, E_{20}, F_4 [5, Fig. 6] also contain two disjoint K-graphs, as observed in [5]. Again, by Corollary 2.9, the independent odd crossing number is nonzero for all of these graphs.

It is also known that each graph B_7, C_4, C_3, D_2 and E_2 [5, Fig. 6] can be obtained from graph A_2 through a sequence of ΔY-exchanges, and the graph E_5 can be obtained from the graph D_3 in the same way [5]. By Lemma 2.3 we need only show that $\mathrm{iocr}_{N_1}(D_3) > 0$ and $\mathrm{iocr}_{N_1}(A_2) > 0$ to prove a nonzero independent odd crossing number for all of these graphs.

Thus we are left with seven graphs, E_{22}, A_2, D_3, F_1, B_1, E_{18}, and E_3. For each we will assume an iocr-0 drawing in the projective plane, then find a contradiction.

Consider E_{22}, letting x be its unique degree 4 vertex as seen in Figure 3. Every 4-cycle not containing x is disjoint from a $K_{2,3}$-graph, so it must be contractible, by Lemma 2.2. Then Lemma 2.6 gives a planar drawing of E_{22}, a contradiction.

We deal with A_2 by a similar argument. Let x be the unique degree 6 vertex in A_2 (see Figure 3). Any triangle in not containing x is disjoint from a K_4-graph and is therefore contractible (Lemma 2.2). Lemma 2.6 gives us the desired result as for E_{22}.

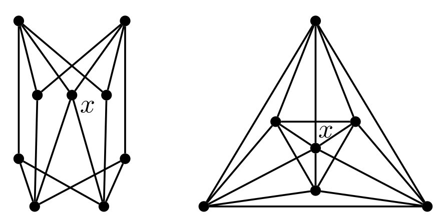

Figure 3: Vertex x in E_{22} (*left*) and in A_2 (*right*)

For graphs F_1 and D_3 we borrow part of an argument from [5]. The cycles $v_1v_2v_3v_4$ and $v_1v_2v_3u_1$ in F_1 (see Figure 4) are each disjoint from a $K_{2,3}$-graph, so they must both be contractible by Lemma 2.2. But the vertices v_1, v_2, v_3, v_4, u_1 induce a $K_{2,3}$-graph in F_1, and at most one of its three cycles is contractible, which contradicts Lemma 2.7.

For D_3 we apply the same argument to its cycles $v_1v_3v_2x$ and $v_1v_3v_2y$ (see Figure 4): Each is disjoint from a K_4-graph so both are contractible (Lemma 2.2). But there is a $K_{2,3}$-graph on vertices v_1, v_2, v_3, x, y, and at most one of its cycles is non-contractible, contradicting Lemma 2.7.

Next consider B_1. We note that, in any drawing of B_1 in the projective plane, any triangle containing exactly one of the vertices x, y, z is contractible since it is disjoint from a K_4-graph. Then by Lemma 2.1, the 4-cycles xu_1yu_2, xu_1zu_2, yu_1zu_2 are all contractible. But these are all the cycles in the $K_{2,3}$-graph on vertices $\{x, y, z\}, \{u_1, u_2\}$, which contradicts Lemma 2.7.

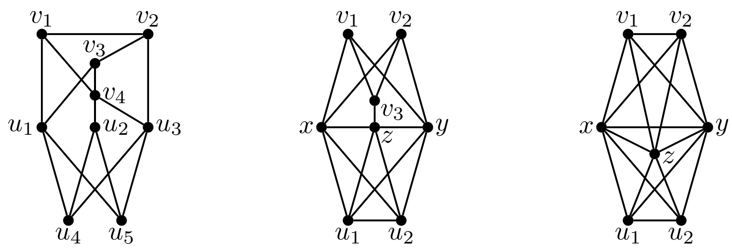

Figure 4: F_1, D_3 and B_1 (left to right) with labels

Only E_{18} and E_3 remain; E_3 requires a lengthier argument which we leave to Section 3.1; so consider E_{18}, which is $K_{4,4}$ with one edge removed (see Figure 5). Suppose $\text{iocr}_{N_1}(E_{18}) = 0$. Let $\{u_1, u_2, u_3, u_4\}$, $\{v_1, v_2, v_3, v_4\}$ be the two partite sets and let u_4v_4 be the missing edge. Each of u_4 and v_4 is contained in 9 induced 4-cycles. Furthermore, for each 4-cycle containing u_4 there is a disjoint 4-cycle containing v_4 [5]. Hence, at least one of the two cycles must be contractible by Corollary 2.9. So one of u_4, v_4, say u_4, belongs to at most 4 non-contractible cycles. But $\{u_4, v_i, v_j, u_i, u_j\}$ induces a $K_{2,3}$-graph for each distinct pair i, j in $\{1, 2, 3\}$, and by Lemma 2.7 each one contains two non-contractible cycles. Since all these cycles are pairwise distinct, there are at least 6 non-contractible 4-cycles containing u_4, a contradiction.

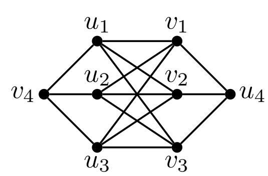

Figure 5: $E_{18} = K_{4,4} - u_4v_4$

3.1 The Forbidden Minor E_3

In this section we show that $\text{iocr}_{N_1}(E_3) > 0$, where $E_3 = K_{3,5}$. Consider an iocr-0 drawing in the projective plane.

Let $\{a_1, a_2, a_3, a_4, a_5\}$ and $\{b_1, b_2, b_3\}$ be the partite sets of $K_{3,5}$. By Corollary 2.8 the drawing contains an induced non-contractible cycle which, without loss of generality, we can assume to be $a_1b_1a_2b_2$. Using Lemma 2.5 we can clear this cycle of all crossings and then cut the surface along it. This creates for each vertex v two new vertices v', v''. The graph is now drawn within a disk with the boundary cycle $a_1'b_1'a_2'b_2'a_1''b_1''a_2''b_2''$ (see Figure 6).

Let G' be the graph drawn within the disk; note that $|E(G')| = |E(G)| - 4$ and for each edge $uv \in E(G)$ with $u \notin \{a_1, b_1, a_2, b_2\}$ and $v \in \{a_1, b_1, a_2, b_2\}$,

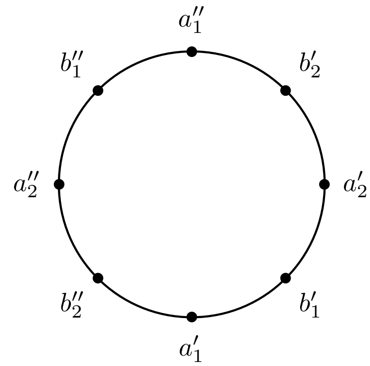

Figure 6: The boundary 8-cycle

we have either uv' or uv'' in $E(G')$.

Lemma 3.1. *G' does not contain two vertex-disjoint paths whose endpoints alternate on the boundary of the disk and that contain at most one vertex from each of the sets $\{v', v''\}$ for $v \in \{a_1, a_2, b_1, b_2\}$.*

Proof. The two paths P_1 and P_2, together with the boundary cycle and a new vertex outside the disk connecting (without crossings) to the four endpoints of the paths, form a K_5-subdivision. Since $\text{iocr}(K_5) > 0$, the K_5-subdivision must contain two non-adjacent subdivided paths that cross an odd number of times; plainly these must be P_1 and P_2. There must be an edge of P_1 and an edge in P_2 that cross oddly. However, we assumed that at most one version of each vertex, either v' or v'' can occur in the paths; hence the odd crossing parity occurs between two edges that were not adjacent in G. This contradicts the assumption that we started with an iocr-0 drawing of G. □

Since b_3a_1 and b_3a_2 are edges of G, there must be corresponding edges in G', and we can assume, without loss of generality, that they are $b_3a'_1$ and $b_3a'_2$ (see Figure 6). Consider the path $b_1a_3b_2$ in G. It cannot be the case that $a_3b'_1 \in E(G')$ since that edge together with either $a_3b'_2$ or $a_3b''_2$ would contradict Lemma 3.1. Hence $a_3b''_1 \in E(G')$, and, by the same argument, $a_4b''_1, a_5b''_1 \in E(G')$.

By the pigeonhole principle at least two of $\{a_3, a_4, a_5\}$ must be adjacent to the same vertex in $\{b'_2, b''_2\}$. Without loss of generality, let us assume that $a_3b'_2$ and $a_4b'_2$ belong to $E(G')$. Let G'' be the graph induced by G' on $\{a_3, a_4, b''_1, b'_2, b_3\}$ together with the edge a'_1b_3, the boundary cycle, and a new vertex outside the disk connected to each of a'_1, b''_1, b'_2 without crossings. Then G'' is a subdivision of $K_{3,3}$ and therefore contains two non-adjacent edges that cross oddly. As in the proof of Lemma 3.1 these two edges must correspond to non-adjacent edges in G that crossed oddly, contradicting the assumption that the drawing of G had independent odd crossing number 0. This completes the proof.

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